Solutions for week 9 exercises

Exercise 5.6.2

The momentum operator is[1] (in the Schrödinger picture)

$$\boldsymbol{P} = -\int d^3x\, \pi(\boldsymbol{x}) \nabla \phi(\boldsymbol{x}). \tag{1}$$

We have

$$\pi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\boldsymbol{p}}}{2}} (a_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{p}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}) \tag{2}$$

$$\phi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} (a_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}) \tag{3}$$

The latter expression gives

$$\nabla \phi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{i\boldsymbol{p}}{\sqrt{2E_{\boldsymbol{p}}}} (a_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{p}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}) \tag{4}$$

Thus

$$\begin{aligned} \boldsymbol{P} &= -\int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} (-i) \sqrt{\frac{E_{\boldsymbol{p}}}{2}} \left[a_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{p}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \right] \frac{i\boldsymbol{q}}{\sqrt{2E_{\boldsymbol{q}}}} \left[a_{\boldsymbol{q}} e^{i\boldsymbol{q}\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{q}} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \right] \\ &= \frac{1}{2} \int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \boldsymbol{q} \sqrt{\frac{E_{\boldsymbol{p}}}{E_{\boldsymbol{q}}}} \Big[-a_{\boldsymbol{p}} a_{\boldsymbol{q}} e^{i(\boldsymbol{p}+\boldsymbol{q})\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} e^{-i(\boldsymbol{p}+\boldsymbol{q})\cdot\boldsymbol{x}} \\ &\quad + a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} e^{i(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{q}} e^{-i(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{x}} \Big] . \end{aligned} \tag{5}$$

Now do the $\boldsymbol{x}$ integration, using

$$\int d^3x\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}} = (2\pi)^3 \delta(\boldsymbol{q}). \tag{6}$$

This gives

$$\boldsymbol{P} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} d^3q\, \boldsymbol{q} \sqrt{\frac{E_{\boldsymbol{p}}}{E_{\boldsymbol{q}}}} \left[-(a_{\boldsymbol{p}} a_{\boldsymbol{q}} + a^\dagger_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}}) \delta(\boldsymbol{p}+\boldsymbol{q}) + (a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} + a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{q}}) \delta(\boldsymbol{p}-\boldsymbol{q}) \right] . \tag{7}$$

[1] Let us recall where this expression comes from. In the classical field theory for a free real scalar field we found that the conserved charge associated with translation invariance in the i direction was

$$P_i = \int d^3x\, \mathcal{T}_i{}^0$$

with $\mathcal{T}_i{}^0 = (\partial^0\phi)(\partial_i\phi) = \pi\, \partial_i\phi$. Up to a sign this is the momentum in the i direction (recall that the momentum $\boldsymbol{P}$ has components P^i, which equals $-P_i$). Thus $\boldsymbol{P} = -\int d^3x\, \pi\nabla\phi$. In the quantized field theory we use the same expression, but with π and $\nabla\phi$ now being operators.

Doing the $\boldsymbol{q}$ integral and using $E_{-\boldsymbol{p}} = E_{\boldsymbol{p}}$ gives

$$\boldsymbol{P} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\boldsymbol{p}\left[a_{\boldsymbol{p}}a_{-\mathbf{p}} + a^\dagger_{\boldsymbol{p}}a^\dagger_{-\mathbf{p}} + a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}\right]. \tag{8}$$

In the first two terms, change integration variable to $\boldsymbol{p}' = -\boldsymbol{p}$ and also change the order of the two operators (which commute with each other). This gives the same expressions as before except that the sign has changed, hence these integrals vanish. Next, change the order of the operators in the third term which introduces an additive constant (from the nonzero commutator) $(2\pi)^3\delta(\mathbf{0})$. The $\boldsymbol{p}$ integration multiplying this constant vanishes since the integrand $\boldsymbol{p}$ is odd. We therefore get

$$\boldsymbol{P} \quad = \quad \int \frac{d^3p}{(2\pi)^3}\,\boldsymbol{p}\;a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}. \tag{9}$$

Remark: The same result is found in the Heisenberg picture, i.e. the momentum operator is conserved (this is the quantum analogue of momentum conservation in the classical field theory). To see this, start from the definition of the Heisenberg picture operator, i.e. $\boldsymbol{P}(t) = e^{iHt}\boldsymbol{P}e^{-iHt}$. Using (9) gives

$$\boldsymbol{P}(t) = \int \frac{d^3p}{(2\pi)^3}\,\boldsymbol{p}\;a^\dagger_{\boldsymbol{p}}(t)a_{\boldsymbol{p}}(t). \tag{10}$$

For the noninteracting Hamiltonian we found the time evolution $a_{\boldsymbol{p}}(t) = a_{\boldsymbol{p}}e^{-iE_{\boldsymbol{p}}t}$. Inserting this and its c.c., we see that the exponentials cancel, giving $\boldsymbol{P}(t) = \boldsymbol{P}$. This result also follows from $[H, \boldsymbol{P}] = 0$ (the calculation of which is very similar to that presented in the exercise below).

Exercise 5.6.3

1) The field operator is given by

$$\psi(\boldsymbol{x}, t) \quad = \quad \int \frac{d^3p}{(2\pi)^3}a_{\boldsymbol{p}}e^{-ipx} \tag{11}$$

where $px \equiv p_\mu x^\mu = p_0 t - \boldsymbol{p}\cdot\boldsymbol{x}$, and $p_0 = \boldsymbol{p}^2/(2m)$. The conserved charge density is (see Exercise 3.6.1) $\rho = j^0 = \psi^*(\boldsymbol{x}, t)\psi(\boldsymbol{x}, t)$. Upon quantization, we find the charge operator

$$\begin{aligned} Q(t) \quad &\equiv \quad \int d^3x\,\psi^\dagger(\boldsymbol{x}, t)\psi(\boldsymbol{x}, t) \\ &= \quad \int d^3x \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}a^\dagger_{\boldsymbol{q}}e^{iqx}a_{\boldsymbol{p}}e^{-ipx}\;. \end{aligned} \tag{12}$$

Integrating over $\boldsymbol{x}$ gives

$$Q(t) \quad = \quad \int \frac{d^3p\,d^3q}{(2\pi)^3}\,a^\dagger_{\boldsymbol{q}}a_{\boldsymbol{p}}\delta(\boldsymbol{p} - \boldsymbol{q})e^{i(q_0 - p_0)t}\;. \tag{13}$$

Integrating over $\boldsymbol{q}$ gives

$$Q(t) = \int \frac{d^3p}{(2\pi)^3}a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}. \tag{14}$$

We see that the rhs is time-independent, i.e. the charge operator is conserved. We can therefore write $Q(t) = Q$. Eq. (14) also shows that the charge operator in this case is the total particle number operator.

2) In the Heisenberg picture operators evolve with time according to $A(t) = e^{iHt} A e^{-iHt}$. Differentiating with respect to time gives Heisenberg's equation of motion:

$$\frac{dA(t)}{dt} = i[H, A(t)]. \tag{15}$$

Thus for the charge operator considered in 1), we have

$$\frac{dQ(t)}{dt} = i[H, Q(t)] = i[H, Q] \tag{16}$$

where the last expression follows from the time independence $Q(t) = Q$ shown in 1), which of course also implies $dQ(t)/dt = 0$. We can therefore conclude that

$$[H, Q] = 0. \tag{17}$$

Next we show (17) by direct calculation. The expression for the Hamiltonian was derived in class:

$$H \quad = \quad \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2m} a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} \; . \tag{18}$$

Thus

$$[H, Q] \quad = \quad \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{p^2}{2m} [N_{\boldsymbol{p}}, N_{\boldsymbol{q}}] \; , \tag{19}$$

where $N_{\boldsymbol{p}} = a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}}$. We have

$$N_{\boldsymbol{q}} N_{\boldsymbol{p}} = a^\dagger_{\boldsymbol{q}} a_{\boldsymbol{q}} a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} \quad = \quad a^\dagger_{\boldsymbol{q}} \left[(2\pi)^3 \delta(\boldsymbol{p} - \boldsymbol{q}) + a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{q}} \right] a_{\boldsymbol{p}} \; , \tag{20}$$

where we used the commutator between $a_{\boldsymbol{q}}$ and $a^\dagger_{\boldsymbol{p}}$. In the second term, we switch the order of the two creation operators, and also of the two annihilation operators (the commutator is zero in each case), giving

$$N_{\boldsymbol{q}} N_{\boldsymbol{p}} \quad = \quad a^\dagger_{\boldsymbol{q}} a_{\boldsymbol{p}} (2\pi)^3 \delta(\boldsymbol{p} - \boldsymbol{q}) + a^\dagger_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} a_{\boldsymbol{p}} a_{\boldsymbol{q}} \; . \tag{21}$$

Finally, using the commutator between $a^\dagger_{\boldsymbol{q}}$ and $a_{\boldsymbol{p}}$ in the second term, we obtain

$$N_{\boldsymbol{q}} N_{\boldsymbol{p}} \quad = \quad a^\dagger_{\boldsymbol{q}} a_{\boldsymbol{p}} (2\pi)^3 \delta(\boldsymbol{p} - \boldsymbol{q}) + a^\dagger_{\boldsymbol{p}} \left[a_{\boldsymbol{p}} a^\dagger_{\boldsymbol{q}} - (2\pi)^3 \delta(\boldsymbol{p} - \boldsymbol{q}) \right] a_{\boldsymbol{q}} \; . \tag{22}$$

Thus

$$N_{\boldsymbol{q}} N_{\boldsymbol{p}} \quad = \quad N_{\boldsymbol{p}} N_{\boldsymbol{q}} + (2\pi)^3 \delta(\boldsymbol{p} - \boldsymbol{q}) \left[a^\dagger_{\boldsymbol{q}} a_{\boldsymbol{p}} - a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{q}} \right] \tag{23}$$

which gives

$$[N_{\boldsymbol{p}}, N_{\boldsymbol{q}}] = (2\pi)^3 \delta(\boldsymbol{p} - \boldsymbol{q}) \left[a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{q}} - a^\dagger_{\boldsymbol{q}} a_{\boldsymbol{p}} . \right] \tag{24}$$

Inserting this into (19) gives

$$\begin{aligned} [H, Q] \quad &= \quad \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{p^2}{2m} (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}) \left[a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{q}} - a^\dagger_{\boldsymbol{q}} a_{\boldsymbol{p}} \right] \\ &= \quad \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2m} \left[a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} - a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}} \right] \\ &= \quad 0. \end{aligned} \tag{25}$$