Solutions for week 10 exercises

Exercise 1

Problem 5.6.5

Let the equation of motion be

$$\mathcal{D}_x \psi(x) = 0 \tag{1}$$

where $\mathcal{D}_x$ is a differential operator that acts on the coordinate x. Then the Green's function $G(x-y)$ is defined as

$$\mathcal{D}_x \, G(x-y) = -i\delta(x-y) \tag{2}$$

where $\delta(x-y)$ is the (4-dimensional) Dirac delta function. The factor $-i$ on the rhs is just a common convention.

The equation of motion for the nonrelativistic field theory considered here is the Schrödinger equation, so $\mathcal{D}_x = -i\partial_0 - \frac{1}{2m}\nabla^2$. Hence the Greens' function satisfies

$$\left[-i\partial_0 - \frac{1}{2m}\nabla^2\right] G(x-y) \quad = \quad -i\delta(x-y)\ . \tag{3}$$

Writing both $G(x-y)$ and $\delta(x-y)$ in terms of their Fourier transform gives

$$\left[-i\partial_0 - \frac{1}{2m}\nabla^2\right] \int \frac{d^4p}{(2\pi)^4} G(p) e^{-ip(x-y)} \quad = \quad -i \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}\ , \tag{4}$$

where, as usual, $p(x-y) \equiv p_\mu(x^\mu - y^\mu) = p_0 x^0 - \vec{p}\cdot(\vec{x}-\vec{y})$. The x-dependence of $G(x-y)$ lies entirely in the exponential factor inside the momentum integral. Moving the differential operator inside the integral and letting it act on this factor gives

$$\int \frac{d^4p}{(2\pi)^4} \left[-p_0 + \frac{\vec{p}^2}{2m}\right] G(p) e^{-ip(x-y)} \quad = \quad -i \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}\ . \tag{5}$$

Comparing the Fourier coefficients on both sides of this equation, the momentum-space propagator $G(p)$ can be read off as

$$G(p) = \frac{i}{p_0 - \frac{\vec{p}^2}{2m}}\ . \tag{6}$$

Problem 5.6.6

In this case the equation of motion is the Dirac equation. Hence $\mathcal{D}_x = i\gamma^\mu \partial_\mu - m$, and therefore the corresponding Green's function (sometimes called the "Dirac propagator") satisfies

$$[i\gamma^\mu \partial_\mu - m]G(x-y) \;=\; -i\delta(x-y)\;. \tag{7}$$

In terms of Fourier transforms this becomes

$$[i\gamma^\mu \partial_\mu - m]\int \frac{d^4p}{(2\pi)^4} G(p) e^{-ip(x-y)} \;=\; -i\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}\;. \tag{8}$$

Differentiating gives

$$\int \frac{d^4p}{(2\pi)^4}[\gamma^\mu p_\mu - m]G(p) e^{-ip(x-y)} \;=\; -i\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}\;. \tag{9}$$

(Here we were careful not to change the order of matrices, as we cannot assume that they commute with each other. To differentiate the exponential, we first moved it left past the matrix $G(p)$ (they commute)). Comparing Fourier coefficients gives

$$[\gamma^\mu p_\mu - m]G(p) = -i. \tag{10}$$

To find $G(p)$ from this matrix equation we can multiply it from the left with the inverse of the matrix $\gamma^\mu p_\mu - m$. However, rather than trying to find this inverse by a direct "brute force" calculation, it is more convenient to multiply the matrix equation with $\gamma^\nu p_\nu + m$ from the left and use that

$$\begin{aligned}(\gamma^\nu p_\nu + m)(\gamma^\mu p_\mu - m) &= \gamma^\nu\gamma^\mu p_\nu p_\mu - m^2 \\ &= \frac{1}{2}\underbrace{\{\gamma^\mu,\gamma^\nu\}}_{2g^{\mu\nu}} p_\mu p_\nu - m^2 = p_\mu p^\mu - m^2 \\ &= p^2 - m^2\;,\end{aligned} \tag{11}$$

which is a c-number. (To go from the second to the third expression we split the first term in two halves and then renamed the dummy summation indices $\mu \leftrightarrow \nu$ in one of them.) Thus

$$(p^2 - m^2)G(p) = -i(\gamma^\nu p_\nu + m), \tag{12}$$

giving

$$G(p) = -i\frac{\gamma^\mu p_\mu + m}{p^2 - m^2}. \tag{13}$$

Note that in the QFT literature one often defines $\gamma^\mu p_\mu \equiv \not{p}$.

One final remark: In these exercises (5.6.5 and 5.6.6) we only made use of the differential equation (2) to find $G(p)$. But to define $G(p)$ uniquely the differential equation is not enough; one also needs to specify boundary conditions. Different propagators (which all satisfy (2)) obeying different boundary conditions can be defined by specifying how the integration contour for the p_0-integral should go around the pole(s) of $G(p)$. This can be implemented by introducing infinitesimal terms $i\epsilon$ (where ϵ is a positive infinitesimal) in the denominator of the propagators. We have seen an example of this in the definition of the so-called Feynman propagator for the real scalar field.

Exercise 2

(a) Let us define the unitary operators

$$\begin{aligned} A &\equiv e^{iH_0(t-t_0)}, & (14)\\ B &\equiv e^{-iH(t-t')}, & (15)\\ C &\equiv e^{-iH_0(t'-t_0)}. & (16)\end{aligned}$$

Then we can write

$$U(t,t') = ABC. \tag{17}$$

Furthermore, we have

$$\begin{aligned} \frac{\partial A}{\partial t} &= iH_0A = iAH_0, & (18)\\ \frac{\partial B}{\partial t} &= -iHB = -iBH, & (19)\\ \frac{\partial C}{\partial t} &= 0. & (20)\end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial}{\partial t}U(t,t') &= \frac{\partial A}{\partial t}BC + A\frac{\partial B}{\partial t}C + AB\frac{\partial C}{\partial t} \\ &= iAH_0BC - iAHBC + 0 \\ &= iA(H_0 - H)BC \\ &= -iAH_{\text{int}}BC \\ &= -iAH_{\text{int}}\underbrace{A^\dagger A}_{=I}BC \\ &= -ie^{iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)}U(t,t') \\ &= -iH_I(t)U(t,t'), & (21)\end{aligned}$$

from which the desired result follows.

It is worth noting that in this calculation we were careful not to interchange the order of (operators involving) H_0 and H, as these do not commute with each other. For the same reason, it follows that, say,

$$AB = e^{iH_0(t-t_0)}e^{-iH(t-t')} \neq e^{iH_0(t-t_0)-iH(t-t')}. \tag{22}$$

More generally, for two arbitrary operators O_1 and O_2,

$$e^{O_1}e^{O_2} \neq e^{O_1+O_2} \quad \text{unless } [O_1, O_2] = 0. \tag{23}$$

This needs to be kept in mind when manipulating exponentials of various operators.[1]

(b) We have

$$\begin{aligned} U^\dagger(t,t')U(t,t') &= (ABC)^\dagger ABC = C^\dagger B^\dagger A^\dagger ABC \\ &= C^\dagger B^\dagger BC = C^\dagger C = I \end{aligned} \tag{24}$$

where we successively used the unitarity of A, B, and C (in that order). We also used that $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$, which is a special case of

$$(O_1 O_2 \ldots O_{n-1} O_n)^\dagger = O_n^\dagger O_{n-1}^\dagger \ldots O_2^\dagger O_1^\dagger. \tag{25}$$

This result can be shown from repeated iteration of the basic property $(O_1 O_2)^\dagger = O_2^\dagger O_1^\dagger$. For example,

$$(O_1 O_2 O_3)^\dagger = ((O_1 O_2) O_3)^\dagger = O_3^\dagger (O_1 O_2)^\dagger = O_3^\dagger O_2^\dagger O_1^\dagger. \tag{26}$$

(c) We have

$$\begin{aligned} U(t_1,t_2)U(t_2,t_3) &= e^{iH_0(t_1-t_0)} e^{-iH(t_1-t_2)} \underbrace{e^{-iH_0(t_2-t_0)} e^{iH_0(t_2-t_0)}}_{=I} e^{-iH(t_2-t_3)} e^{-iH_0(t_3-t_0)} \\ &= e^{iH_0(t_1-t_0)} \underbrace{e^{-iH(t_1-t_2)} e^{-iH(t_2-t_3)}}_{=e^{-iH(t_1-t_2)-iH(t_2-t_3)}} e^{-iH_0(t_3-t_0)} \\ &= e^{iH_0(t_1-t_0)} e^{-iH(t_1-t_3)} e^{-iH_0(t_3-t_0)} \\ &= U(t_1,t_3). \end{aligned} \tag{27}$$

Note that in the second line here we were able to combine the two indicated exponentials into one only because the arguments of the two exponentials commuted in this case. Similarly,

$$\begin{aligned} U(t_1,t_3)U^\dagger(t_2,t_3) &= e^{iH_0(t_1-t_0)} e^{-iH(t_1-t_3)} e^{-iH_0(t_3-t_0)} e^{iH_0(t_3-t_0)} e^{iH(t_2-t_3)} e^{-iH_0(t_2-t_0)} \\ &= e^{iH_0(t_1-t_0)} e^{-iH(t_1-t_3)} e^{iH(t_2-t_3)} e^{-iH_0(t_2-t_0)} \\ &= e^{iH_0(t_1-t_0)} e^{-iH(t_1-t_2)} e^{-iH_0(t_2-t_0)} \\ &= U(t_1,t_2). \end{aligned} \tag{28}$$

[1] If $[O_1, O_2]$ commutes with both O_1 and O_2 (as happens e.g. if $[O_1, O_2]$ is a c-number), one has the simple formula $e^{O_1}e^{O_2} = e^{O_1+O_2}e^{[O_1,O_2]/2}$. Otherwise things are very complicated (see the section "The Zassenhaus formula" in the Wikipedia article "Baker-Campbell-Hausdorff formula").