Solution for week 11 exercises

Exercise 1

(a) The diagram has 2 vertices (internal points with 4 lines attached) and is therefore a 2nd order diagram.

(b) Labeling the left vertex as z_1 and the right vertex as z_2, the Feynman rules give the following expression for the diagram:

$$\frac{(-i\lambda)^2}{S} \int d^4 z_1 \int d^4 z_2 \; D_F(x - z_1)[D_F(z_1 - z_2)]^3 D_F(z_2 - y). \tag{1}$$

(c) As discussed in the lectures, the perturbation expansion for the 2-point function is given by the sum of all connected diagrams. Expressions for these diagrams can be deduced from the numerator in the expression for the perturbation expansion, which reads

$$\langle 0|T\{\phi_I(x)\phi_I(y)\exp\left[-i\int dt H_I(t)\right]\}|0\rangle. \tag{2}$$

As $H_I(t) \propto \lambda$, the term $\propto \lambda^2$ is given by the 2nd order term in the expansion of the exponential:

$$\frac{1}{2!}\langle 0|T\{\phi_I(x)\phi_I(y)(-i)^2\int dt_1 \int dt_2 H_I(t_1)H_I(t_2)\}|0\rangle. \tag{3}$$

Inserting

$$H_I(t) = \frac{\lambda}{4!}\int d^3 z[\phi_I(t, \vec{z})]^4 \tag{4}$$

this can be written

$$\frac{1}{2!}\left(\frac{-i\lambda}{4!}\right)^2 \int d^4 z_1 \int d^4 z_2 \; \langle 0|T\{\phi_I(x)\phi_I(y)\phi_I(z_1)\phi_I(z_1)\phi_I(z_1)\phi_I(z_1)\phi_I(z_2)\phi_I(z_2)\phi_I(z_2)\phi_I(z_2)\}|0\rangle, \tag{5}$$

where we defined $z_i^\mu = (t_i, \vec{z}_i)$ $(i = 1, 2)$. From the structure of the diagram one sees that it should represent pairings of the following type: Pair $\phi_I(x)$ with one of the 4 fields at one of the vertices, and pair $\phi_I(y)$ with one of the 4 fields at the other vertex, and furthermore, pair the 3 remaining fields at one vertex with the 3 remaining fields at the other vertex. This gives a combinatorial factor

$$2 \cdot 4 \cdot 4 \cdot 3! \tag{6}$$

One factor of 4 comes from the 4 different ways to pair $\phi_I(x)$ with one of the 4 fields of a given vertex, and the other factor of 4 comes from the 4 different ways to pair $\phi_I(y)$ with one of the 4 fields at the other vertex. The factor of 2 comes from the fact that the "given vertex" could be either z_1 or z_2 (making the "other vertex" z_2 or z_1, respectively), corresponding to the two possible ways of labeling the vertices in the diagram. The factor of 3! arises because this is the number of ways to pair the 3 remaining fields at one vertex with the 3 remaining fields at the other. Thus we get

$$\frac{1}{S} = \frac{2 \cdot 4 \cdot 4 \cdot 3!}{2!(4!)^2} = \frac{1}{6} \quad \Rightarrow \quad S = 6. \tag{7}$$

Exercise 2 (= Exercise 5.6.7)

1) For the Dirac Lagrangian density

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi, \tag{8}$$

we found in the exercises for week 6 that the Hamiltonian density is

$$\mathcal{H} \;\; = \;\; -i\bar{\psi}\gamma^j \partial_j \psi + m\bar{\psi}\psi \tag{9}$$

and the 4-current is

$$j^\mu \;\; = \;\; \bar{\psi}\gamma^\mu \psi.$$

This gives

$$j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi \tag{10}$$

and thus

$$\mathcal{K} \;\; \equiv \;\; \mathcal{H} - \mu j^0 = -i\bar{\psi}\gamma^j\partial_j\psi + m\bar{\psi}\psi - \mu\psi^\dagger\psi. \tag{11}$$

2) We interpret the quantity $\mathcal{K}$ as the Hamiltonian density $\mathcal{H}'$ associated with a Lagrangian density $\mathcal{L}'$:

$$\mathcal{H}' = \Pi'_\psi \partial_0 \psi - \mathcal{L}' \tag{12}$$

where

$$\Pi'_\psi = \frac{\partial \mathcal{L}'}{\partial(\partial_0 \psi)}. \tag{13}$$

We want to find $\mathcal{L}'$. Let us use the ansatz ("guess") that

$$\mathcal{L}' = \mathcal{L} + f(\bar{\psi}, \psi), \tag{14}$$

in particular, f is not a function of $\partial_0\psi$ or $\partial_0\bar{\psi}$. Thus

$$\Pi'_\psi = \frac{\partial \mathcal{L}'}{\partial(\partial_0\psi)} = \frac{\partial \mathcal{L}}{\partial(\partial_0\psi)} + \underbrace{\frac{\partial f}{\partial(\partial_0\psi)}}_{=0} = \frac{\partial \mathcal{L}}{\partial(\partial_0\psi)} = \Pi_\psi \tag{15}$$

(Note also that because neither f nor $\mathcal{L}$ depend on $\partial_0\bar{\psi}$, there is no term $(\partial_0\bar{\psi})\Pi'_{\bar{\psi}}$ on the rhs of (12).) This gives

$$f = \mathcal{L}' - \mathcal{L} = (\Pi'_\psi\partial_0\psi - \mathcal{H}') - (\Pi_\psi\partial_0\psi - \mathcal{H}) = \mathcal{H} - \mathcal{H}' = \mu j^0 = \mu\bar{\psi}\gamma^0\psi. \tag{16}$$

As the rhs is only a function of $\bar{\psi}$ and ψ, our ansatz was correct. Thus

$$\begin{aligned} \mathcal{L}' &= \mathcal{L} + f = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi + \mu\bar{\psi}\gamma^0\psi \\ &= i\bar{\psi}\gamma^0(\partial_0 - i\mu)\psi + i\bar{\psi}\gamma^j\partial_j\psi - m\bar{\psi}\psi. \end{aligned} \tag{17}$$

We see that $\mathcal{L}'$ can be obtained from $\mathcal{L}$ by the replacement $\partial_0 \to \partial_0 - i\mu$, as claimed in the problem text.

3) The Lagrangian density for a complex scalar field is

$$\mathcal{L} \;=\; (\partial_\mu\Phi)^*\,(\partial^\mu\Phi) - m^2\Phi^*\Phi\;. \tag{18}$$

We have

$$\Pi_\Phi \;=\; \frac{\partial\mathcal{L}}{\partial(\partial_0\Phi)} = \partial^0\Phi^*\;,$$

and similarly $\Pi_{\Phi^*} = \partial^0\Phi$. The Hamiltonian density then becomes

$$\begin{aligned} \mathcal{H} &= \Pi_\Phi\dot{\Phi} + \Pi_{\Phi^*}\dot{\Phi}^* - \mathcal{L} \\ &= 2\Pi_{\Phi^*}\Pi_\Phi - \mathcal{L} \\ &= \Pi_{\Phi^*}\Pi_\Phi + (\nabla\Phi)^*\cdot(\nabla\Phi) + m^2\Phi^*\Phi\;. \end{aligned} \tag{19}$$

The conserved current is given by Eq. (3.42) in JOA's lecture notes (see also week 4 exercises):

$$j^\mu \;=\; i\left[\Phi\,(\partial^\mu\Phi)^* - \Phi^*\,(\partial^\mu\Phi)\right]. \tag{20}$$

The quantity $\mathcal{K} = \mathcal{H} - \mu j^0$ is to be interpreted as a Hamiltonian-type density. Thus we must express j^0 in terms of Π_Φ and Π_Φ^* instead of in terms of the time derivatives of Φ and Φ^*. This gives

$$j^0 \;=\; i\left[\Pi_\Phi\Phi - \Pi_{\Phi^*}\Phi^*\right]\;, \tag{21}$$

and so

$$\mathcal{K} \;=\; \mathcal{H} - \mu j^0 = \Pi_{\Phi^*}\Pi_\Phi + (\nabla\Phi)^*\cdot(\nabla\Phi) + m^2\Phi^*\Phi \;-\mu i\left[\Pi_\Phi\Phi - \Pi_{\Phi^*}\Phi^*\right].$$

We can construct the associated Lagrangian density $\mathcal{L}'$ from

$$\mathcal{L}' = \Pi_\Phi \dot{\Phi} + \Pi_{\Phi^*} \dot{\Phi}^* - \mathcal{K} \tag{22}$$

where $\dot{\Phi}$ is given by Hamilton's equation

$$\dot{\Phi} \quad = \quad \frac{\partial \mathcal{K}}{\partial \Pi_\Phi} = \Pi_{\Phi^*} - i\mu\Phi \; .$$

Thus

$$\Pi_{\Phi^*} = (\partial_0 + i\mu)\Phi. \tag{23}$$

Similarly, from Hamilton's equation $\dot{\Phi}^* = \partial\mathcal{K}/\partial\Pi_{\Phi^*}$ one obtains $\Pi_\Phi = (\partial_0 - i\mu)\Phi^*$. One now uses these equations to eliminate Π_Φ and Π_{Φ^*} from the rhs of (22), as appropriate for a Lagrangian-type density. This gives

$$\mathcal{L}' \quad = \quad [(\partial_0 + i\mu)\Phi]^*[(\partial^0 + i\mu)\Phi] + (\partial_i\Phi)^*(\partial^i\Phi) - m^2\Phi^*\Phi. \tag{24}$$

Two comments:

- Note that if we had reversed the overall sign in the expression for j^μ in Eq. (20) (which would have been an equally good choice since the conservation law $\partial_\mu j^\mu = 0$ would still hold), the sign of the $i\mu$ terms in $\mathcal{L}'$ would have been reversed as well. (Of course, the same thing is true for the problem of the Dirac field considered in 1) and 2)).

- Note that in this problem of a complex scalar field the difference $\mathcal{L}' - \mathcal{L}$ is found to be not just a function of Φ and Φ^*, but also depends on their time derivatives. Thus an ansatz $\mathcal{L}' = \mathcal{L} + g(\Phi, \Phi^*)$ analogous to what we used for the Dirac field would have failed for this problem.