TFY4210, Quantum theory of many-particle systems, 2015: Tutorial 1

1. Many-particle wavefunctions.

Consider an arbitrary Hermitian single-particle operator $\hat{O}$. In the "first quantization" formalism, $\hat{O}$ takes the form

$$\hat{O} = \sum_{i=1}^{N} \hat{o}_i \tag{1}$$

where N is the number of particles in the system. Let $\phi_\nu(x)$ and o_ν be the eigenfunctions and associated eigenvalues of $\hat{o}$, i.e.

$$\hat{o}\,\phi_\nu(x) = o_\nu \phi_\nu(x). \tag{2}$$

Here ν is a set of quantum numbers which completely characterize the single-particle eigenfunctions. As discussed in the lectures, we can use these to construct a basis set for many-particle wavefunctions. The (normalized) basis functions can be written

$$\Phi_{\nu_1,\nu_2,\ldots,\nu_N}(x_1, x_2, \ldots, x_N) = \frac{1}{\sqrt{N!}\sqrt{\prod_\nu n_\nu!}} \sum_{P \in S_N} \xi^{t_P} \cdot P\phi_{\nu_1}(x_1)\phi_{\nu_2}(x_2)\ldots\phi_{\nu_N}(x_N) \tag{3}$$

where $\xi = \pm 1$ for bosons/fermions and t_P is the number of transpositions (2-particle permutations) associated with the permutation P.[1] S_N is the set of all $N!$ permutations. Furthermore, n_ν is the number of particles in the single-particle state ϕ_ν in the many-particle state $\Phi_{\nu_1,\nu_2,\ldots,\nu_N}$ (for fermions this can only be 0 or 1, hence $\sqrt{\prod_\nu n_\nu!} = 1$ in the fermionic case and can therefore be omitted).

(a) Write down an example of a basis function for a system of 3 fermions where all single-particle states ν_1, ν_2, ν_3 are different (write the state out explicitly, i.e. all $3! = 6$ terms).

(b) Demonstrate that the function changes sign if the coordinates of two particles are interchanged, e.g. x_1 and x_2. Furthermore, demonstrate that if two or more of the single-particle states are chosen to be the same, the basis function vanishes. These properties are in accordance with the Pauli exclusion principle.

(c) Show that $\Phi_{\nu_1,\nu_2,\ldots,\nu_N}$ as given in (3) is an eigenfunction of $\hat{O}$ with eigenvalue $\sum_\nu o_\nu n_\nu$.

[1] As discussed in the lectures, t_P is not unique, but its evenness/oddness is, so that the sign ξ^{t_P} is well-defined.

2. Fermionic creation and annihilation operators.

(a) Calculate $c_3 c_2^\dagger |1_1, 0_2, 1_3, \ldots\rangle$.

(b) Use the definitions of the operators $c_\nu^\dagger$ and c_ν to show that these operators satisfy the defining property of adjoint operators, i.e.

$$\langle \bar{n} | c_\nu | n \rangle = \langle n | c_\nu^\dagger | \bar{n} \rangle^*. \tag{4}$$

Here $|n\rangle \equiv |n_1, n_2, \ldots\rangle$ and $|\bar{n}\rangle \equiv |\bar{n}_1, \bar{n}_2, \ldots\rangle$ are two arbitrary basis states for a fermionic many-particle system.

(c) Show that the fermionic creation and annihilation operators satisfy

$$\{c_\mu, c_\nu^\dagger\} = \delta_{\mu,\nu}. \tag{5}$$

In your proof, treat the two cases $\mu = \nu$ and $\mu \neq \nu$ separately (for the latter case you may limit your analysis to $\mu < \nu$ for simplicity).

(d) Use the fermionic anti-commutation relations to show that the fermionic number operator $\hat{n}_\nu \equiv c_\nu^\dagger c_\nu$ satisfies

$$\hat{n}_\nu^2 = \hat{n}_\nu. \tag{6}$$

Use this result to deduce the possible eigenvalues of $\hat{n}_\nu$.

3. Some useful commutator expressions.

(a) Show that, for arbitrary operators A, B, and C,

$$[AB, C] = A[B, C]_\zeta - \zeta [A, C]_\zeta B, \tag{7}$$

where $[A, B]_\zeta \equiv AB + \zeta BA$, with $\zeta = \mp 1$ corresponding to the commutator and anti-commutator, respectively.

(b) Show that, for both bosonic and fermionic creation and annihilation operators,

$$[\hat{n}_\mu, c_\nu^\dagger] = \delta_{\mu,\nu}\, c_\nu^\dagger. \tag{8}$$

$$[\hat{n}_\mu, c_\nu] = -\delta_{\mu,\nu}\, c_\nu. \tag{9}$$