TFY4210, Quantum theory of many-particle systems, 2014: Tutorial 6

1. The Holstein-Primakoff representation.

(a) Show that an alternative and equivalent form of the spin commutation relations

$$[S^x, S^y] = iS^z, \quad [S^y, S^z] = iS^x, \quad [S^z, S^x] = iS^y \tag{1}$$

is given by

$$[S^+, S^-] = 2S^z, \quad [S^z, S^\pm] = \pm S^\pm. \tag{2}$$

(b) The Holstein-Primakoff (HP) representation is given by

$$S^+ = \sqrt{2S - a^\dagger a}\, a, \tag{3}$$

$$S^- = a^\dagger \sqrt{2S - a^\dagger a}, \tag{4}$$

$$S^z = S - a^\dagger a, \tag{5}$$

where a and $a^\dagger$ are canonical boson operators. Show that the HP representation satisfies the correct spin commutation relations and the relation $\boldsymbol{S} \cdot \boldsymbol{S} = S(S+1)$. NB! For this you should use the exact HP expressions; do NOT expand the square roots. Hint: Use that $[f(\hat{O}), g(\hat{O})] = 0$ for functions f and g of an operator $\hat{O}$.

2. Ferromagnetic Heisenberg model with a spin anisotropy.

Consider spins on a two-dimensional square lattice with Hamiltonian

$$H = J \sum_{\langle i,j \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j - D \sum_i (S_i^z)^2. \tag{6}$$

The sum in the first term is over all pairs of nearest-neighbour sites, and the sum in the second term is over all sites. The exchange constant $J = -|J| < 0$.

(a) Assuming that the parameter $D > 0$, argue that the spins will order along the $+z$ or $-z$ direction.

(b) Use spin-wave theory to calculate the ground state energy E_0 and the magnon dispersion $\omega_{\boldsymbol{k}}$.

(c) The energy gap is defined as $\Delta \equiv E_1 - E_0$ where E_1 is the energy of the lowest excited state. Give an expression for Δ.

(d) What do you predict about the ordering direction if $D < 0$, and why?

3. Physical picture of ferromagnetic spin waves

Consider a Heisenberg ferromagnet which has magnetic order with the magnetization vector pointing in the z direction. Let the state $|\boldsymbol{k}\rangle \equiv a^{\dagger}_{\boldsymbol{k}}|0\rangle$ where $|0\rangle$ is the ferromagnetic ground state, i.e. the state $|\boldsymbol{k}\rangle$ contains one magnon with wavevector $\boldsymbol{k}$ ($\neq 0$). Let us define the transverse correlation function in the state $|\boldsymbol{k}\rangle$ as

$$\langle \boldsymbol{k}|\boldsymbol{S}^{\perp}_i \cdot \boldsymbol{S}^{\perp}_j|\boldsymbol{k}\rangle, \tag{7}$$

where the transverse spin operator $\boldsymbol{S}^{\perp}_i$ is the projection of the spin operator onto the xy plane:

$$\boldsymbol{S}^{\perp}_i = S^x_i \hat{x} + S^y_i \hat{y}. \tag{8}$$

Use the truncated HP representation to show that

$$\langle \boldsymbol{k}|\boldsymbol{S}^{\perp}_i \cdot \boldsymbol{S}^{\perp}_j|\boldsymbol{k}\rangle = \frac{2S}{N}\cos[\boldsymbol{k}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)]. \tag{9}$$

Thus on average each spin has a small transverse component (i.e. perpendicular to the direction of the magnetization) and the orientations of the transverse components of two spins i and j differ by an angle $\boldsymbol{k}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)$. This should hopefully make the origin of the terminology *spin wave* clearer.