

# TFY4210, Quantum theory of many-particle systems, 2015:

## Tutorial 6

### 1. The Holstein-Primakoff representation.

(a) Show that an alternative and equivalent form of the spin commutation relations

$$[S^x, S^y] = iS^z, \quad [S^y, S^z] = iS^x, \quad [S^z, S^x] = iS^y \quad (1)$$

is given by

$$[S^+, S^-] = 2S^z, \quad [S^z, S^\pm] = \pm S^\pm. \quad (2)$$

(b) The Holstein-Primakoff (HP) representation is given by

$$S^+ = \sqrt{2S - a^\dagger a} a, \quad (3)$$

$$S^- = a^\dagger \sqrt{2S - a^\dagger a}, \quad (4)$$

$$S^z = S - a^\dagger a, \quad (5)$$

where  $a$  and  $a^\dagger$  are canonical boson operators. Show that the HP representation satisfies the correct spin commutation relations and the relation  $\mathbf{S} \cdot \mathbf{S} = S(S+1)$ . NB! For this you should use the exact HP expressions; do NOT expand the square roots. Hint: Use that  $[f(\hat{O}), g(\hat{O})] = 0$  for functions  $f$  and  $g$  of an operator  $\hat{O}$ .

### 2. Ferromagnetic Heisenberg model with a spin anisotropy.

Consider spins on a two-dimensional square lattice with Hamiltonian

$$H = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j - D \sum_i (S_i^z)^2. \quad (6)$$

The sum in the first term is over all pairs of nearest-neighbour sites, and the sum in the second term is over all sites. The exchange constant  $J = -|J| < 0$ .

(a) Assuming that the parameter  $D > 0$ , argue that the spins will order along the  $+z$  or  $-z$  direction.

(b) Use spin-wave theory to calculate the ground state energy  $E_0$  and the magnon dispersion  $\omega_{\mathbf{k}}$ .

(c) The energy gap is defined as  $\Delta \equiv E_1 - E_0$  where  $E_1$  is the energy of the lowest excited state. Give an expression for  $\Delta$ .

(d) What do you predict about the ordering direction if  $D < 0$ , and why?

### 3. Physical picture of ferromagnetic spin waves

Consider a Heisenberg ferromagnet which has magnetic order with the magnetization vector pointing in the  $z$  direction. Let the state  $|\mathbf{k}\rangle \equiv a_{\mathbf{k}}^\dagger|0\rangle$  where  $|0\rangle$  is the ferromagnetic ground state, i.e. the state  $|\mathbf{k}\rangle$  contains one magnon with wavevector  $\mathbf{k}$  ( $\neq 0$ ). Let us define the transverse correlation function in the state  $|\mathbf{k}\rangle$  as

$$\langle \mathbf{k} | \mathbf{S}_i^\perp \cdot \mathbf{S}_j^\perp | \mathbf{k} \rangle, \quad (7)$$

where the transverse spin operator  $\mathbf{S}_i^\perp$  is the projection of the spin operator onto the  $xy$  plane:

$$\mathbf{S}_i^\perp = S_i^x \hat{x} + S_i^y \hat{y}. \quad (8)$$

Use the truncated HP representation to show that

$$\langle \mathbf{k} | \mathbf{S}_i^\perp \cdot \mathbf{S}_j^\perp | \mathbf{k} \rangle = \frac{2S}{N} \cos[\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)]. \quad (9)$$

Thus on average each spin has a small transverse component (i.e. perpendicular to the direction of the magnetization) and the orientations of the transverse components of two spins  $i$  and  $j$  differ by an angle  $\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)$ . This should hopefully make the origin of the terminology *spin wave* clearer.