TFY4210, Quantum theory of many-particle systems, 2014: Tutorial 8

All problems deal with the Heisenberg nearest-neighbour antiferromagnet on a hypercubic lattice in d dimensions, which is analyzed using spin-wave theory in which interactions between the Holstein-Primakoff bosons are neglected.

1. Eigenstates and eigenvalues of the total S^z operator.

Consider the operator $S^z = \sum_j S_j^z$ representing the total spin in the z direction (the sum is over all N sites in the system).

(a) Show that

$$S^z = \sum_{\boldsymbol{k}} (\beta_{\boldsymbol{k}}^\dagger \beta_{\boldsymbol{k}} - \alpha_{\boldsymbol{k}}^\dagger \alpha_{\boldsymbol{k}}). \tag{1}$$

(b) Argue that a general eigenstate of the Heisenberg antiferromagnet in the spin-wave approximation can be written (we omit normalization factors)

$$\prod_{\boldsymbol{k}} [(\alpha_{\boldsymbol{k}}^\dagger)^{n_{\alpha_{\boldsymbol{k}}}} (\beta_{\boldsymbol{k}}^\dagger)^{n_{\beta_{\boldsymbol{k}}}}] |G\rangle \tag{2}$$

where $\{n_{\alpha_{\boldsymbol{k}}}\}$ and $\{n_{\beta_k}\}$ are the magnon numbers characterizing the eigenstate and $|G\rangle$ is the ground state discussed in the lectures. Show that this state is also an eigenstate of S^z and determine the corresponding eigenvalue.

2. Transverse spin correlation functions.

In this problem we consider the transverse spin correlations in the ground state $|G\rangle$ and in the single-magnon states $\alpha_{\boldsymbol{k}}^\dagger|G\rangle \equiv |\boldsymbol{k}^{(\alpha)}\rangle$ and $\beta_{\boldsymbol{k}}^\dagger|G\rangle \equiv |\boldsymbol{k}^{(\beta)}\rangle$. The transverse spin correlation functions for the respective states are given by the expectation value of $\boldsymbol{S}_i^\perp \cdot \boldsymbol{S}_j^\perp$ where $\boldsymbol{S}_i^\perp = S_i^x \hat{x} + S_i^y \hat{y}$ is the transverse spin operator on site i (introduced in Problem 3 in Tutorial 6). The two sites i and j can either be on the same sublattice or on different sublattices; consider both cases.

(a) Find $\langle G|\boldsymbol{S}_i^\perp \cdot \boldsymbol{S}_j^\perp|G\rangle$.

(b) Find $\langle \boldsymbol{k}^{\alpha)}|\boldsymbol{S}_i^\perp \cdot \boldsymbol{S}_j^\perp|\boldsymbol{k}^{(\alpha)}\rangle$ and $\langle \boldsymbol{k}^{\beta)}|\boldsymbol{S}_i^\perp \cdot \boldsymbol{S}_j^\perp|\boldsymbol{k}^{(\beta)}\rangle$.

(The final expressions will involve single wavevector sums; do not try to evaluate these.)

3. The temperature-dependent part of the sublattice magnetization correction.

(a) The dispersion relation for antiferromagnetic magnons is $\omega_{\boldsymbol{k}} = JSz\sqrt{1-\gamma_{\boldsymbol{k}}^2}$ where $\gamma_{\boldsymbol{k}} = (2/z)\sum_{\boldsymbol{\delta}} \cos(\boldsymbol{k}\cdot\boldsymbol{\delta})$ and $z = 2d$ is the number of nearest neighbour sites. Show that for small $|\boldsymbol{k}|$,

$$\gamma_{\boldsymbol{k}} \approx 1 - \frac{\boldsymbol{k}^2}{2d}, \tag{3}$$

$$\omega_{\boldsymbol{k}} \approx 2JS\sqrt{d}|\boldsymbol{k}|. \tag{4}$$

We found that the temperature-dependent part of the sublattice magnetization correction ΔM_A is given by

$$\frac{2}{N}\sum_{\boldsymbol{k}} n_{\boldsymbol{k}} \frac{1}{\sqrt{1-\gamma_{\boldsymbol{k}}^2}} \tag{5}$$

where $n_{\boldsymbol{k}} = \frac{1}{e^{\omega_{\boldsymbol{k}}/k_B T}-1}$.

(b) Show that in $d = 1$ and $d = 2$ the contribution from small $\boldsymbol{k}$ makes (5) diverge. Hence there is no magnetic order in the Heisenberg antiferromagnet at finite temperatures in one and two dimensions.

(c) In contrast, show that in $d = 3$ the expression (5) is finite (i.e. there is no divergence at small $\boldsymbol{k}$) and scales as T^2.