1. Relationship between $G^<(\nu,\omega)$ and the spectral function $A(\nu,\omega)$

(a) $$G^<(\nu, t-t') = i \langle c_\nu^\dagger(t') c_\nu(t) \rangle$$

(for fermions and $\nu = \nu'$)

$$= i \frac{1}{Z} \mathrm{Tr}\left(e^{-\beta H} e^{iHt'} c_\nu^\dagger e^{-iHt'} e^{iHt} c_\nu e^{-iHt} \right)$$

Evaluate trace using eigenstates of H as basis: (inserting $I = \sum_m |m\rangle\langle m|$ twice)

$G^<(\nu, t-t')$

$$= \frac{i}{Z} \sum_{m,n} \langle m| e^{-\beta H} e^{iHt'} c_\nu^\dagger e^{-iHt'} |n\rangle \langle n| e^{iHt} c_\nu e^{-iHt} |m\rangle$$

$$= \frac{i}{Z} \sum_{m,n} e^{-\beta E_m} e^{iE_m t'} e^{-iE_n t'} \langle m| c_\nu^\dagger |n\rangle e^{iE_n t} e^{-iE_m t} \langle n| c_\nu |m\rangle$$

$$= \frac{i}{Z} \sum_{m,n} e^{-\beta E_m} e^{i(E_n - E_m)(t-t')} |\langle m| c_\nu^\dagger |n\rangle|^2$$

The Fourier transform is

$$G^<(\nu,\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, G^<(\nu,t)$$

$$= \frac{i}{Z}\sum_{n,m} e^{-\beta E_m} |\langle m|c_\nu^\dagger|n\rangle|^2 \cdot \underbrace{\int_{-\infty}^{\infty} dt\, e^{i(\omega+E_n-E_m)t}}_{2\pi\,\delta(\omega+E_n-E_m)}$$

$$= \frac{2\pi i}{Z}\sum_{n,m} e^{-\beta E_m} |\langle m|c_\nu^\dagger|n\rangle|^2\, \delta(\omega+E_n-E_m)$$

(b) $A(\nu,\omega)$ is given by [Eq. (53) in notes]

$$A(\nu,\omega) = \frac{1}{Z}\sum_{nm} |\langle m|c_\nu^\dagger|n\rangle|^2 \left(e^{-\beta E_n} + e^{-\beta E_m}\right) \cdot \delta(\omega+E_n-E_m)$$

The δ-function gives $E_n = E_m - \omega$

$$\Rightarrow e^{-\beta E_n} + e^{-\beta E_m} = e^{-\beta E_m}\left(1 + e^{\beta\omega}\right)$$

Because $(1+e^{\beta\omega})$ is indep. of m,n it can be taken outside the n,m sums

$$\Rightarrow \quad A(\nu,\omega) = \left(1+e^{\beta\omega}\right)\frac{1}{Z} \cdot \sum_{n,m} |\langle m|c_\nu^\dagger|n\rangle|^2 \, e^{-\beta E_m} \, \delta(\omega+E_n-E_m)$$

(c) This gives, by comparing the expressions for $G^<(\nu,\omega)$ and $A(\nu,\omega)$, that

$$+\, i \cdot 2\pi \cdot \underbrace{\left(1+e^{\beta\omega}\right)^{-1}}_{n_F(\omega)} A(\nu,\omega) = G^<(\nu,\omega)$$

2. An alternative form of the Lehmann representation.

Consider the spectral function as given by Eq. (53) in the notes,

$$A(\nu,\omega) = \frac{1}{Z}\sum_{n,m} |\langle m|c_\nu^\dagger|n\rangle|^2 \left(e^{-\beta E_n} + e^{-\beta E_m}\right) \cdot \delta(\omega + E_n - E_m)$$

This gives

$$\int_{-\infty}^{\infty} d\omega' \frac{A(\nu,\omega')}{\omega-\omega'+i\eta}$$

$$= \frac{1}{Z}\sum_{n,m} |\langle m|c_\nu^\dagger|n\rangle|^2 \left(e^{-\beta E_n} + e^{-\beta E_m}\right) \cdot \underbrace{\int_{-\infty}^{\infty} d\omega' \frac{\delta(\omega'+E_n-E_m)}{\omega-\omega'+i\eta}}_{\frac{1}{\omega-(E_m-E_n)+i\eta}}$$

$$= \frac{1}{Z}\sum_{n,m} \frac{|\langle m|c_\nu^\dagger|n\rangle|^2}{\omega+E_n-E_m+i\eta}\left(e^{-\beta E_n} + e^{-\beta E_m}\right),$$

which we recognize as the Lehmann representation for $G^R(\nu,\omega)$, Eq. (50) in the notes. QED

4. Calculating $G^R(\nu,t)$ from $G^R(\nu,\omega)$ by contour integration.

We have

$$G^R(\nu,\omega) = \frac{1}{Z}\sum_{n,m} \frac{|\langle m|c_\nu^\dagger|n\rangle|^2}{\omega+E_n-E_m+i\eta}\left(e^{-\beta E_n}+e^{-\beta E_m}\right)$$

Want to calculate

$$G^R(\nu,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\, G^R(\nu,\omega)$$

$$= \frac{1}{Z} \sum_{n,m} |\langle m | c_\nu^\dagger | n \rangle|^2 \left(e^{-\beta E_n} + e^{-\beta E_m} \right)$$

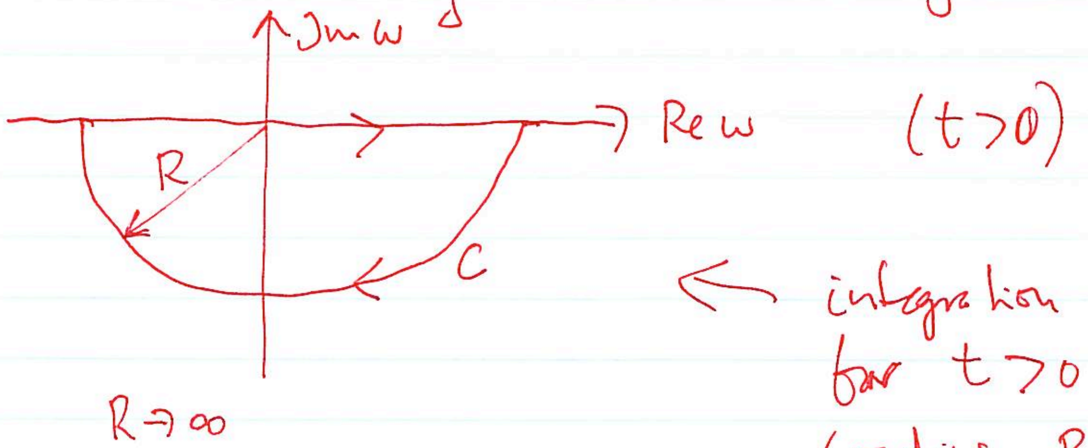
$$\cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega t}}{\omega + E_n - E_m + i\eta}$$

We will do the ω-integral using contour integration. The contour will be taken to be a semicircle in the upper or lower half plane, together with the integral over the real axis. For complex ω we have

$$e^{-i\omega t} = e^{-i(\mathrm{Re}\,\omega + i\,\mathrm{Im}\,\omega)t}$$

$$= e^{-it\,\mathrm{Re}\,\omega} \, e^{t\,\mathrm{Im}\,\omega}$$

For $t > 0$, $e^{t\,\mathrm{Im}\,\omega}$ will lead to decay if $\mathrm{Im}\,\omega < 0$ $\Rightarrow$ semicircle must be in the lower half plane for semicircle part of integral to vanish as radius of semicircle goes to ∞:



$\leftarrow$ integration contour for $t > 0$ (radius $R \to \infty$)

Calculate integral using residue theorem.
The integrand has a pole at

$$\omega = E_m - E_n - i\eta$$

i.e. in the lower half plane, <u>inside</u> the integration contour C.

Thus we get, for $t > 0$,

$$\int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega t}}{\omega + E_n - E_m + i\eta}$$

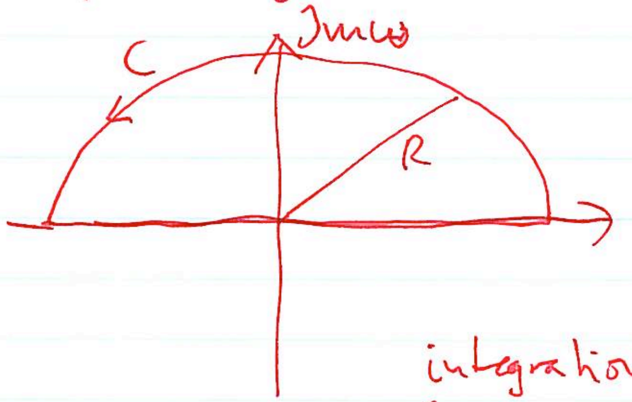
$$= \oint_C d\omega \, \frac{e^{-i\omega t}}{\omega + E_n - E_m + i\eta}$$

$$= 2\pi i \cdot \underbrace{(-1)}_{\substack{\text{because}\\ \text{contour}\\ \text{is clockwise}}} \cdot \operatorname*{Res}_{\omega = E_m - E_n - i\eta} \frac{e^{-i\omega t}}{\omega + E_n - E_m - i\eta}$$

$$= -2\pi i \cdot e^{-i(E_m - E_n - i\eta)t}$$

$$\xrightarrow[\eta \to 0]{} \; -2\pi i \; e^{i(E_n - E_m)t}$$

On the other hand, for $t < 0$ the contour must be closed in the upper half plane $\mathrm{Im}\,\omega > 0$ since only then does $e^{t\,\mathrm{Im}\,\omega}$ cause the necessary convergence of the semicircle part of the integral:



$(t < 0)$

integration contour for $t < 0$
(radius $R \to \infty$)

Now there are no poles inside the contour, so the residue theorem gives that the integral vanishes. Thus

$$G^R(\nu,t) = \frac{1}{Z} \sum_{n,m} |\langle m| c_\nu^\dagger |n\rangle|^2$$

$$\cdot \left(e^{-\beta E_n} + e^{-\beta E_m}\right) \cdot \frac{1}{2\pi} \cdot (-2\pi i)\, e^{i(E_n - E_m)t}\, \theta(t)$$

$$= -i\,\theta(t) \frac{1}{Z} \sum_{n,m} \left(e^{-\beta E_n} + e^{-\beta E_m}\right)$$

$$\cdot\, e^{i(E_n - E_m)t}\, |\langle m| c_\nu^\dagger |n\rangle|^2$$

<u>QED.</u>

4. Fermi liquids.

(a) The spectral function of a Fermi liquid was given in the notes [Eq. (60)]:

$$A_g(\boldsymbol{k}\sigma,\omega) = Z_{\boldsymbol{k}} \cdot \frac{1}{\pi}\frac{(1/2\tau_{\boldsymbol{k}})}{(\omega-\xi^*_{\boldsymbol{k}})^2+(1/2\tau_{\boldsymbol{k}})^2} + A_{\text{incoherent}}(\boldsymbol{k}\sigma,\omega). \tag{1}$$

Here we have introduced the subscript g to stand for "general", to indicate that this is the most general form of the spectral function for a Fermi liquid. On the other hand, the Fermi liquid spectral function given in the problem text is

$$A_s(\boldsymbol{k}\sigma,\omega) = Z\cdot\delta(\omega-\xi_{\boldsymbol{k}}) + (1-Z)\frac{\theta(W-|\omega|)}{2W}, \tag{2}$$

where the introduced subscript s stands for "simplified", as this expression is the spectral function associated with a simplified model of a Fermi liquid. We have the following correspondences between factors in the two expressions:

- $Z_{\boldsymbol{k}} \Rightarrow Z$. Thus in the simplified model, the $\boldsymbol{k}$-dependence of $Z_{\boldsymbol{k}}$ is neglected.
- $\frac{1}{\pi}\frac{(1/2\tau_{\boldsymbol{k}})}{(\omega-\xi^*_{\boldsymbol{k}})^2+(1/2\tau_{\boldsymbol{k}})^2} \Rightarrow \delta(\omega-\xi_{\boldsymbol{k}})$. Thus in the simplified model, the finite lifetime $\tau_{\boldsymbol{k}}$ of the quasi-particles has been neglected, i.e. the quasi-particles have been taken to have an infinite lifetime. Also the renormalization (due to interactions) of the quasi-particle dispersion from $\xi_{\boldsymbol{k}}$ to some other function $\xi^*_{\boldsymbol{k}}$ has been neglected.
- $A_{\text{incoherent}}(\boldsymbol{k}\sigma,\omega) \Rightarrow (1-Z)\frac{\theta(W-|\omega|)}{2W}$. Thus in the simplified model a simple function that is step-wise constant has been used to represent the "incoherent" background.

(b)

$$\int_{-\infty}^{\infty} d\omega\ A_s(\boldsymbol{k}\sigma,\omega) \;=\; Z\underbrace{\int_{-\infty}^{\infty} d\omega\ \delta(\omega-\xi_{\boldsymbol{k}})}_{=1} + \frac{1-Z}{2W}\underbrace{\int_{-\infty}^{\infty} d\omega\ \theta(W-|\omega|)}_{=\int_{-W}^{W}d\omega=2W} = 1. \tag{3}$$

(c) From Eq. (62) in the notes we have that the momentum distribution function of the Fermi liquid described by the spectral function A_s is given by

$$\langle c^\dagger_{\boldsymbol{k}\sigma}c_{\boldsymbol{k}\sigma}\rangle = \int_{-\infty}^{\infty} d\omega\ A_s(\boldsymbol{k}\sigma,\omega)n_F(\omega). \tag{4}$$

At zero temperature, $n_F(\omega)=\theta(-\omega)$, giving

$$\langle c^\dagger_{\boldsymbol{k}\sigma}c_{\boldsymbol{k}\sigma}\rangle = \int_{-\infty}^{\infty} d\omega\ A_s(\boldsymbol{k}\sigma,\omega)\theta(-\omega). \tag{5}$$

Inserting for A_s and doing the ω integration gives

$$\langle c^\dagger_{\boldsymbol{k}\sigma}c_{\boldsymbol{k}\sigma}\rangle = Z\theta(-\xi_{\boldsymbol{k}}) + \frac{1}{2}(1-Z). \tag{6}$$

From this expression one sees that when $|\boldsymbol{k}|$ crosses the Fermi surface from above, i.e. when $\xi_{\boldsymbol{k}}$ changes sign from positive to negative, the momentum distribution function jumps by Z. Note that in the non-interacting case, $Z=1$, this expression for the momentum distribution function reduces to the (zero-temperature) Fermi-Dirac distribution $n_F(\xi_{\boldsymbol{k}})$, for which the jump is 1 at the Fermi surface.