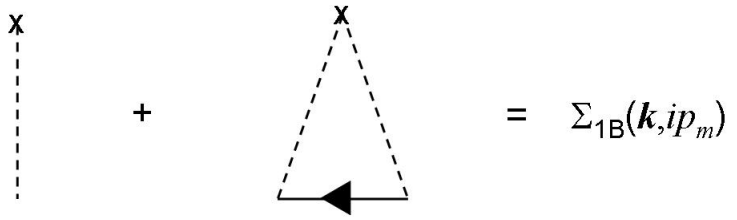
TFY4210, Quantum theory of many-particle systems, 2015: Tutorial 11

1. The full Born approximation.

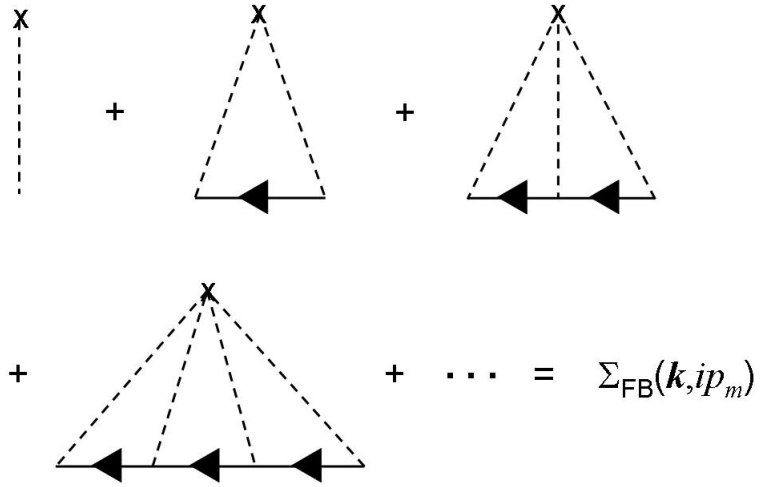
(c) In the lectures we showed that $\bar{\mathcal{G}}(\boldsymbol{k}, ip_m)$ can be expressed as

$$\bar{\mathcal{G}}(\boldsymbol{k}, ip_m) = \frac{1}{(\mathcal{G}^{(0)}(\boldsymbol{k}, ip_m))^{-1} - \Sigma(\boldsymbol{k}, ip_m)} = \frac{1}{ip_m - \xi_{\boldsymbol{k}} - \Sigma(\boldsymbol{k}, ip_m)} \tag{1}$$

where $\Sigma(\boldsymbol{k}, ip_m)$ is the self-energy, defined as the sum of all self-energy diagrams. We found an approximate result for $\bar{\mathcal{G}}(\boldsymbol{k}, ip_m)$ by approximating the self-energy $\Sigma(\boldsymbol{k}, ip_m)$ as



This is called the "first Born approximation," which is why we have here defined the sum of these two self-energy diagrams as $\Sigma_{1\mathrm{B}}(\boldsymbol{k}, ip_m)$. In the following we will study a different approximation to the self-energy, given by the (infinite) sum of *all* self-energy diagrams with a *single* impurity cross:



This is called the "full Born approximation," which is why we have defined this sum as $\Sigma_{\mathrm{FB}}(\boldsymbol{k}, ip_m)$. Note that the first two terms in this infinite sum constitute the "first Born

approximation" $\Sigma_{\rm 1B}(\boldsymbol{k}, ip_m)$.

(a) Explain under what physical conditions (specified by the impurity density $n_{\rm imp}$ and the strength of the scattering potential $U(\boldsymbol{r})$) you expect $\Sigma_{\rm 1B}$ to be a good approximation to the full self-energy. Do the same for $\Sigma_{\rm FB}$.

(b) Give the mathematical expression for the nth diagram in $\Sigma_{\rm FB}(\boldsymbol{k}, ip_m)$, i.e. the diagram with n interaction lines where n is an arbitrary positive integer. (You may find it helpful to first consider the expressions for the first few diagrams in $\Sigma_{\rm FB}(\boldsymbol{k}, ip_m)$ before you consider the case of a general n.)

(c) Assume that the impurity potential $U(\boldsymbol{r})$ is very short-ranged, so that its Fourier transform $U(\boldsymbol{k})$ can be approximated to be independent of $\boldsymbol{k}$, i.e. set $U(\boldsymbol{k}) \equiv U$ for all $\boldsymbol{k}$. Show that in this case, $\Sigma_{\rm FB}(\boldsymbol{k}, ip_m)$ is given by

$$\Sigma_{\rm FB}(\boldsymbol{k}, ip_m) = \frac{NU}{1 - U\sum_{\boldsymbol{k}_1} \mathcal{G}^{(0)}(\boldsymbol{k}_1, ip_m)} \equiv \Sigma_{\rm FB}(ip_m). \tag{2}$$

(d) In the lectures we found that for a very short-ranged impurity potential, the second diagram in $\Sigma_{\rm 1B}$ and $\Sigma_{\rm FB}$ in the figures above is given by

$$-\frac{i}{2\tau_{\rm 1B}}\mathrm{sgn}(p_m) \quad \text{where} \quad \frac{1}{\tau_{\rm 1B}} = 2\pi n_{\rm imp} u^2 D(0). \tag{3}$$

(In the lectures $\tau_{\rm 1B}$ was just called τ.) Here $u = U\Omega$ and $D(0)$ is a density-of-states factor whose precise definition is unimportant here. Use the result (3) to show that

$$\mathrm{Im}\,\Sigma_{\rm FB}(ip_m) = -\frac{1}{2\tau_{\rm FB}}\mathrm{sgn}(p_m) \tag{4}$$

and give an expression for the parameter $\tau_{\rm FB}$.

(e) Determine the spectral function $\bar{A}(\boldsymbol{k}, \omega) \equiv -(1/\pi)\mathrm{Im}\,\overline{G^R}(\boldsymbol{k}, \omega)$ where $\overline{G^R}(\boldsymbol{k}, \omega)$ is the impurity-averaged retarded Green function. Express its maximum value and its width at half the maximum in terms of $\tau_{\rm FB}$, and show that it satisfies the sum rule $\int_{-\infty}^{\infty} d\omega\, \bar{A}(\boldsymbol{k}, \omega) = 1$.

2. An alternative perturbation expansion for the Green function for the impurity scattering problem.

(In order to simplify the notation, we will in this problem not explicitly write the wavevector and Matsubara frequency that various Green functions and self-energy diagrams depend on. So instead of $\mathcal{G}^{(0)}(\boldsymbol{k}, ip_m)$ we simply write $\mathcal{G}^{(0)}$, and similarly for other quantities.)

The self-energy Σ is the sum of all self-energy diagrams. Let $\Sigma^{(1)}$ be the self-energy diagram with a single impurity cross and a single interaction line, as shown in Fig. 1.

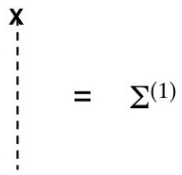


Figure 1: The self-energy diagram $\Sigma^{(1)}$.

Define Σ' to be the sum of all the other self-energy diagrams. Thus

$$\Sigma = \Sigma^{(1)} + \Sigma'. \tag{5}$$

The self-energy diagrams that make up Σ' fall into two different classes. One class consists of those diagrams that contain no factors of $\Sigma^{(1)}$. Let us call such diagrams **bare** diagrams. The other class consists of those diagrams that contain at least one factor of $\Sigma^{(1)}$. Let us call such diagrams **tree** diagrams. We choose the names "bare" and "tree" because each $\Sigma^{(1)}$ resembles a tree (see Fig. 4 below); a diagram with no trees is bare. The process of changing a self-energy diagram by "planting a tree" corresponds to replacing a $\mathcal{G}^{(0)}$ with $\mathcal{G}^{(0)}\Sigma^{(1)}\mathcal{G}^{(0)}$ (with identical wavevectors for the $\mathcal{G}^{(0)}$'s in both expressions). This is illustrated in Fig. 2. Thus the diagrams before and after the replacement differ by a factor $\Sigma^{(1)}\mathcal{G}^{(0)}$.

Figure 2: The replacement process involved in "planting a tree". Left figure: Before the planting. Right figure: After the planting.

By taking a tree diagram and removing all factors of $\Sigma^{(1)}\mathcal{G}^{(0)}$ we get a bare diagram. Thus to any tree diagram we can associate a unique bare diagram. Conversely, any tree diagram can be "generated" by starting from its associated bare diagram and inserting factors of $\Sigma^{(1)}\mathcal{G}^{(0)}$. These things are probably best understood diagrammatically: see Figs. 3 and 4.

In the following we use the subscript α to label bare self-energy diagrams. Consider an arbitrary bare self-energy diagram $\Sigma_{(\alpha)}$.

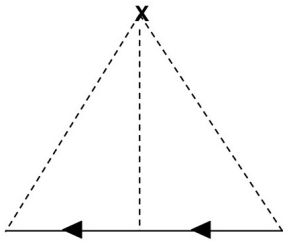


Figure 3: An example of a "bare" self-energy diagram.

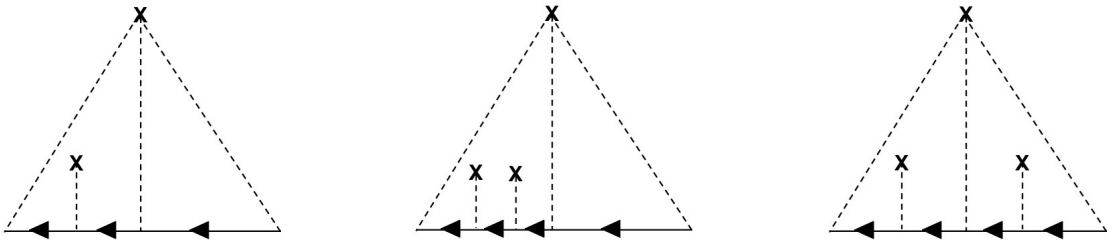


Figure 4: A few of the "tree" self-energy diagrams that can be generated from the bare diagram in Fig. 3 by inserting factors of $\Sigma^{(1)}\mathcal{G}^{(0)}$.

Define

$$\begin{aligned}\widetilde{\Sigma}_{(\alpha)} &= \Sigma_{(\alpha)} + \text{ the sum of all tree self-energy diagrams that} \\ &\quad \text{can be generated from } \Sigma_{(\alpha)} \text{ by "planting trees".}\end{aligned} \tag{6}$$

(a) Show that the mathematical expression for $\widetilde{\Sigma}_{(\alpha)}$ can be obtained from the mathematical expression of the bare diagram $\Sigma_{(\alpha)}$ by replacing every $\mathcal{G}^{(0)}$ in that expression by another Green function $\widetilde{\mathcal{G}}^{(0)}$ given by

$$\widetilde{\mathcal{G}}^{(0)} = \frac{1}{(\mathcal{G}^{(0)})^{-1} - \Sigma^{(1)}}. \tag{7}$$

(b) Show that

$$\Sigma' = \widetilde{\Sigma} \tag{8}$$

where

$$\widetilde{\Sigma} \equiv \sum_{\alpha} \widetilde{\Sigma}_{(\alpha)}. \tag{9}$$

(c) Show that $\bar{\mathcal{G}}$ can be expressed in terms of $\widetilde{\mathcal{G}}^{(0)}$ and $\widetilde{\Sigma}$ as

$$\bar{\mathcal{G}} = \frac{1}{(\widetilde{\mathcal{G}}^{0})^{-1} - \widetilde{\Sigma}}. \tag{10}$$

(d) Next, observe that Eq. (10) can be interpreted as the result of summing an alternative perturbation expansion for $\bar{\mathcal{G}}$ in which $\widetilde{\mathcal{G}}^{(0)}$ plays the role of an unperturbed Green function and the self-energy diagrams are the $\widetilde{\Sigma}_{(\alpha)}$'s. We can represent each term in this expansion as a Feynman diagram. We represent $\widetilde{\mathcal{G}}^{(0)}$ diagrammatically by a **thick** full line. Furthermore,

given the result of (a), the diagram for $\widetilde{\Sigma}_{(\alpha)}$ is obtained from that for $\Sigma_{(\alpha)}$ by replacing the thin electron lines ($\mathcal{G}^{(0)}$) by thick ones ($\widetilde{\mathcal{G}}^{(0)}$). An example of a Feynman diagram in this expansion is shown in Fig. 5.

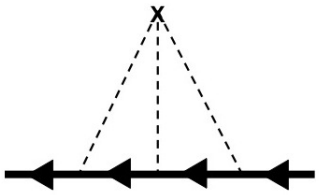


Figure 5: Example of Feynman diagram in the alternative perturbation expansion.

Draw all Feynman diagrams in this alternative perturbation expansion for $\bar{\mathcal{G}}$ up to and including order $n = 4$ in the electron-impurity scattering potential $U(\boldsymbol{q})$. (You don't need to write the wavevectors for the various lines.) What would you consider to be an advantage of this expansion compared to the original one discussed in the lectures?