TFY4210, Quantum theory of many-particle systems, 2015: Tutorial 12

1. Current density operators and linear response description of coupling a system of electrons to an external electric field.

This problem deals with the proof of some results that were stated and used in the discussion in the lecture notes leading up to the proof of the Kubo formula for the electric conductivity [more specifically, the relevant equations in the lecture notes are Eqs. (170)-(177)].

(a) The current density operator can be written $\boldsymbol{j}(\boldsymbol{r}) = \boldsymbol{j}^P(\boldsymbol{r}) + \boldsymbol{j}^D(\boldsymbol{r})$ where, in first quantization,

$$\boldsymbol{j}^P(\boldsymbol{r}) = -\frac{ie}{2m}\sum_i [\nabla_i \delta(\boldsymbol{r}-\boldsymbol{r}_i) + \delta(\boldsymbol{r}-\boldsymbol{r}_i)\nabla_i], \tag{1}$$

$$\boldsymbol{j}^D(\boldsymbol{r}) = -\frac{e^2}{m}\sum_i \boldsymbol{A}(\boldsymbol{r}_i)\delta(\boldsymbol{r}-\boldsymbol{r}_i). \tag{2}$$

Show that the second-quantized expressions for these operators are

$$\boldsymbol{j}^P(\boldsymbol{r}) = -\frac{ie}{2m}\sum_\sigma \left(\psi_\sigma^\dagger(\boldsymbol{r})\nabla\psi_\sigma(\boldsymbol{r}) - [\nabla\psi_\sigma^\dagger(\boldsymbol{r})]\psi_\sigma(\boldsymbol{r})\right), \tag{3}$$

$$\boldsymbol{j}^D(\boldsymbol{r}) = -\frac{e^2}{m}\boldsymbol{A}(\boldsymbol{r})\sum_\sigma \psi_\sigma^\dagger(\boldsymbol{r})\psi_\sigma(\boldsymbol{r}), \tag{4}$$

where $\psi_\sigma^\dagger(\boldsymbol{r})$ and $\psi_\sigma(\boldsymbol{r})$ respectively create and annihilate an electron with spin projection σ at position $\boldsymbol{r}$.

(b) The vector potential enters the Hamiltonian through the term which in first quantization reads

$$\sum_i \frac{1}{2m}(-i\nabla_i - e\boldsymbol{A}(\boldsymbol{r}_i))^2. \tag{5}$$

Show that the corresponding expression in second quantization can be written

$$\frac{1}{2m}\sum_\sigma \int d\boldsymbol{r}\ \psi_\sigma^\dagger(\boldsymbol{r})(-i\nabla - e\boldsymbol{A}(\boldsymbol{r}))^2\psi_\sigma(\boldsymbol{r}). \tag{6}$$

(c) The linear response coupling H_{ext} can be extracted as the part of (6) that is linear in $\boldsymbol{A}$. Show that this gives

$$H_{\text{ext}} = -\int d\boldsymbol{r}\boldsymbol{j}^P(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r}). \tag{7}$$

2. Dirac equation for $m = 0$.

Dirac's linearization problem

$$\sqrt{p_x^2 + p_y^2 + p_z^2 + m^2c^2} = \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta mc \tag{8}$$

simplifies if $m = 0$. Show that one can then use a 2-component wavefunction Ψ which satisfies

$$\frac{1}{c}\frac{\partial \Psi}{\partial t} = -\boldsymbol{\sigma} \cdot \nabla \Psi, \tag{9}$$

where the components of the vector $\boldsymbol{\sigma}$ are the Pauli matrices. (This equation is called the Weyl equation.)

3. Alternative matrix representations.

In the lectures we discussed the standard ("Dirac-Pauli") representation of the γ-matrices in the Dirac equation. However, other, physically equivalent, representations are also possible. Consider a valid representation $\{\gamma^\mu\}$ (e.g. the Dirac-Pauli one) and define

$$\tilde{\gamma}^\mu = S\gamma^\mu S^\dagger \tag{10}$$

where S is a unitary 4×4 matrix. In this problem you will show that $\{\tilde{\gamma}^\mu\}$ is also a valid representation which gives physically equivalent results to the representation $\{\gamma^\mu\}$.

(a) Show that the matrices $\{\tilde{\gamma}^\mu\}$ satisfy the appropriate anticommutation relations, i.e. for all μ, ν,

$$\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2g^{\mu\nu}. \tag{11}$$

(b) Let Ψ be a solution of the Dirac equation expressed in terms of the $\{\gamma^\mu\}$ representation, i.e.

$$(i\hbar\gamma^\mu\partial_\mu - mc)\Psi = 0. \tag{12}$$

Show that the corresponding solution $\tilde{\Psi}$ of the Dirac equation expressed in terms of the $\{\tilde{\gamma}^\mu\}$ representation, i.e.

$$(i\hbar\tilde{\gamma}^\mu\partial_\mu - mc)\tilde{\Psi} = 0, \tag{13}$$

is given by $\tilde{\Psi} = S\Psi$.

(c) The probability density and probability current density affect physical quantities and must therefore be the same for the representations $\{\gamma^\mu\}$ and $\{\tilde{\gamma}^\mu\}$. Show that this is indeed the case, i.e. show that

$$\tilde{\rho} = \rho \quad \text{and} \quad \tilde{\boldsymbol{j}} = \boldsymbol{j}. \tag{14}$$

4. Eigenvalues of $\alpha_k\alpha_l$ $(k \neq l)$.

Deduce the eigenvalues of the product $\alpha_k\alpha_l$ for $k \neq l$, which appears in the components of the spin operator for Dirac particles. (Solve the problem by making use of algebraic properties of the α matrices, avoid using an explicit matrix representation.)