

TFY4210, Quantum theory of many-particle systems, 2016: Solution to tutorial 12

1. The full Born approximation.

(a) In both Σ_{1B} and Σ_{FB} , the self-energy diagrams are limited to having one impurity cross, i.e. $\propto n_{\text{imp}}$, so we expect the validity to be limited to sufficiently low impurity densities. Furthermore, the self-energy diagrams in Σ_{1B} are of low order (respectively of 1st and 2nd order) in the scattering potential U , so we expect the validity of Σ_{1B} to be limited to sufficiently weak scattering. Σ_{FB} can be expected to be valid for stronger scattering than Σ_{1B} as it contains self-energy diagrams of arbitrarily high order in the scattering potential.

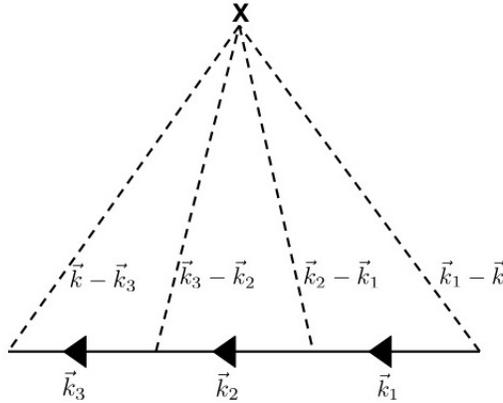
(b) The n 'th diagram has n factors of the scattering potential and $n - 1$ Green functions, with $n - 1$ internal wavenumbers that are summed over. The expression for this diagram can be written

$$N \sum_{\mathbf{k}_1, \dots, \mathbf{k}_{n-1}} U(\mathbf{k}_1 - \mathbf{k}) U(\mathbf{k}_2 - \mathbf{k}_1) \cdots U(\mathbf{k} - \mathbf{k}_{n-1}) \mathcal{G}^{(0)}(\mathbf{k}_1) \mathcal{G}^{(0)}(\mathbf{k}_2) \cdots \mathcal{G}^{(0)}(\mathbf{k}_{n-1}). \quad (1)$$

Alternatively, a more explicit way of writing it is

$$N \sum_{\mathbf{k}_1, \dots, \mathbf{k}_{n-1}} U(\mathbf{k}_1 - \mathbf{k}) \mathcal{G}^{(0)}(\mathbf{k}_1) \left[\prod_{i=2}^{n-1} U(\mathbf{k}_i - \mathbf{k}_{i-1}) \mathcal{G}^{(0)}(\mathbf{k}_i) \right] U(\mathbf{k} - \mathbf{k}_{n-1}). \quad (2)$$

As a concrete example, the $n = 4$ diagram is shown below.



(c) Again consider the n 'th diagram. If the \mathbf{k} -dependence of the scattering potential can be neglected, the dependence on the scattering potential simplifies to a constant U^n . The wavevector summations then simplify to $n - 1$ identical summations over a single wavevector. To get Σ_{FB} we sum over all diagrams, i.e. over n from $n = 1$ to ∞ :

$$\Sigma_{FB} = N \sum_{n=1}^{\infty} U^n \left(\sum_{\mathbf{k}_1} \mathcal{G}^{(0)}(\mathbf{k}_1) \right)^{n-1} = NU \sum_{n=0}^{\infty} \left(U \sum_{\mathbf{k}_1} \mathcal{G}^{(0)}(\mathbf{k}_1) \right)^n. \quad (3)$$

This is a geometric series. It can be evaluated either by using the (given) result for the sum of such a series, or by noting that it can be written in terms of itself as $NU + \Sigma_{\text{FB}}U \sum_{\mathbf{k}_1} \mathcal{G}^{(0)}(\mathbf{k}_1)$ and solving for Σ_{FB} . This gives (we do not address questions about convergence here)

$$\Sigma_{\text{FB}} = \frac{NU}{1 - U \sum_{\mathbf{k}_1} \mathcal{G}^{(0)}(\mathbf{k}_1)}. \quad (4)$$

(d) Using the Feynman rules, the second diagram in Σ_{FB} is given by

$$NU^2 \sum_{\mathbf{k}_1} \mathcal{G}^{(0)}(\mathbf{k}_1) = n_{\text{imp}}u \cdot U \sum_{\mathbf{k}_1} \mathcal{G}^{(0)}(\mathbf{k}_1). \quad (5)$$

Thus, using the information given in the text about the expression for this diagram, we get (reinstating the Matsubara frequency dependence of $\mathcal{G}^{(0)}$, which we have suppressed in the notation so far)

$$U \sum_{\mathbf{k}_1} \mathcal{G}^{(0)}(\mathbf{k}_1, ip_m) = \frac{1}{n_{\text{imp}}u} \left(-\frac{i}{2\tau_{1\text{B}}} \text{sgn}(p_m) \right) = -i\pi u D(0) \text{sgn}(p_m). \quad (6)$$

Inserting this into the expression for Σ_{FB} gives

$$\Sigma_{\text{FB}}(ip_m) = \frac{n_{\text{imp}}u}{1 + i\pi u D(0) \text{sgn}(p_m)} = n_{\text{imp}}u \frac{1 - i\pi u D(0) \text{sgn}(p_m)}{1 + (\pi u D(0))^2}. \quad (7)$$

The imaginary part is

$$\text{Im } \Sigma_{\text{FB}}(ip_m) = -\frac{\pi n_{\text{imp}}u^2 D(0)}{1 + (\pi u D(0))^2} \text{sgn}(p_m) \equiv -\frac{1}{2\tau_{\text{FB}}} \text{sgn}(p_m). \quad (8)$$

Therefore

$$\tau_{\text{FB}} = \frac{1 + (\pi u D(0))^2}{2\pi n_{\text{imp}}u^2 D(0)}. \quad (9)$$

This result can alternatively be written in the form $\tau_{\text{FB}} = \tau_{1\text{B}} \left(1 + \frac{1}{(2\tau_{1\text{B}}NU)^2} \right)$, which also could have been found without invoking the explicit result $1/\tau_{1\text{B}} = 2\pi n_{\text{imp}}u^2 D(0)$.

(e) Except for simply replacing the first Born approximation results for the self-energy with those of the full Born approximation, the problem of finding the spectral function is identical to that discussed in the lecture notes (Sec. 4.8, with the result for the spectral function given in Eq. (158)). Thus in Eq. (158) we must replace $n_{\text{imp}}u \rightarrow n_{\text{imp}}u/(1 + (\pi u D(0))^2)$ in the real part of the self-energy (cf. Eq. (7)) and $\tau \rightarrow \tau_{\text{FB}}$ in the imaginary part. For other aspects of the calculation we refer to the lecture notes. Defining $\xi_{\mathbf{k}}^* \equiv \xi_{\mathbf{k}} + n_{\text{imp}}u/(1 + (\pi u D(0))^2)$, we get

$$\bar{A}(\mathbf{k}, \omega) = \frac{1}{\pi} \frac{1/2\tau_{\text{FB}}}{(\omega - \xi_{\mathbf{k}}^*)^2 + (1/2\tau_{\text{FB}})^2}. \quad (10)$$

Its maximum value is

$$\bar{A}(\mathbf{k}, \omega = \xi_{\mathbf{k}}^*) = \frac{2\tau_{\text{FB}}}{\pi}. \quad (11)$$

At half the maximum we must have

$$(\omega - \xi_{\mathbf{k}}^*)^2 = \frac{1}{(2\tau_{\text{FB}})^2}. \quad (12)$$

Therefore the width of the function at half the maximum is

$$\Delta\omega = 2 \cdot \frac{1}{2\tau_{\text{FB}}} = \frac{1}{\tau_{\text{FB}}}. \quad (13)$$

Furthermore, any Lorentzian function integrates to 1:

$$\int_{-\infty}^{\infty} dx \frac{1}{\pi} \frac{b}{(x - c)^2 + b^2} = 1 \quad (14)$$

for any real c and positive real b . As (10) takes the form of a Lorentzian with $x = \omega$, $b = 1/2\tau_{\text{FB}}$ and $c = \xi_{\mathbf{k}}^*$, this proves the sum rule.

2. An alternative perturbation expansion for the Green function for the impurity scattering problem.

(a) Suppose that the bare self-energy diagram $\Sigma_{(\alpha)}$ has r Green function lines with wavevectors $\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_r$ (these wavevectors are not necessarily all distinct, but that's unimportant here). The mathematical expression for $\Sigma_{(\alpha)}$ is then

$$\Sigma_{(\alpha)} = \dots \mathcal{G}^{(0)}(\mathbf{k}_1) \mathcal{G}^{(0)}(\mathbf{k}_2) \dots \mathcal{G}^{(0)}(\mathbf{k}_r) \quad (15)$$

where we have suppressed everything in the expression (including wavevector summations) except the Green function factors. From this bare diagram one can generate tree diagrams with m_1 trees on the first Green function line, m_2 trees on the second, ..., and m_r trees on the r th Green function line. Thus in a tree diagram the integers m_1, m_2, \dots, m_r are nonnegative, and they are not all 0 (the bare diagram corresponds to all being 0). To get $\tilde{\Sigma}_{(\alpha)}$ we should add all the tree diagrams to the bare diagram. Therefore

$$\begin{aligned} \tilde{\Sigma}_{(\alpha)} = & \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \dots \sum_{m_r=0}^{\infty} \dots [\mathcal{G}^{(0)}(\mathbf{k}_1)(\Sigma^{(1)}\mathcal{G}^{(0)}(\mathbf{k}_1))^{m_1}] [\mathcal{G}^{(0)}(\mathbf{k}_2)(\Sigma^{(1)}\mathcal{G}^{(0)}(\mathbf{k}_2))^{m_2}] \dots \\ & [\mathcal{G}^{(0)}(\mathbf{k}_r)(\Sigma^{(1)}\mathcal{G}^{(0)}(\mathbf{k}_r))^{m_r}]. \end{aligned} \quad (16)$$

Compared to the expression in (15) we have simply replaced each factor $\mathcal{G}^{(0)}(\mathbf{k}_i)$ with $\mathcal{G}^{(0)}(\mathbf{k}_i)(\Sigma^{(1)}\mathcal{G}^{(0)}(\mathbf{k}_i))^{m_i}$ and then we are summing over each m_i from 0 to ∞ . The sums are independent of each other, and each sum is a geometric series. Using the expression for the sum of a geometric series, $\sum_{m=0}^{\infty} x^m = 1/(1-x)$, we get

$$\mathcal{G}^{(0)}(\mathbf{k}_i) \sum_{m_i=0}^{\infty} (\mathcal{G}^{(0)}(\mathbf{k}_i)\Sigma^{(1)})^{m_i} = \mathcal{G}^{(0)}(\mathbf{k}_i) \frac{1}{1 - \mathcal{G}^{(0)}(\mathbf{k}_i)\Sigma^{(1)}} = \frac{1}{(\mathcal{G}^{(0)}(\mathbf{k}_i))^{-1} - \Sigma^{(1)}} = \tilde{\mathcal{G}}^{(0)}(\mathbf{k}_i). \quad (17)$$

Therefore

$$\tilde{\Sigma}_{(\alpha)} = \dots \tilde{\mathcal{G}}^{(0)}(\mathbf{k}_1) \tilde{\mathcal{G}}^{(0)}(\mathbf{k}_2) \dots \tilde{\mathcal{G}}^{(0)}(\mathbf{k}_r). \quad (18)$$

We see that this expression is identical in form to (15), but each $\mathcal{G}^{(0)}$ in (15) has been replaced by $\tilde{\mathcal{G}}^{(0)}$, which is what we wanted to show.

(b) All self-energy diagrams except $\Sigma^{(1)}$ are included exactly once in the sum $\sum_{\alpha} \tilde{\Sigma}_{(\alpha)} \equiv \tilde{\Sigma}$. That is, in $\tilde{\Sigma}$ we are not missing any of the self-energy diagrams (except $\Sigma^{(1)}$) and we are not including any of them more than once. Therefore $\Sigma' = \tilde{\Sigma}$.

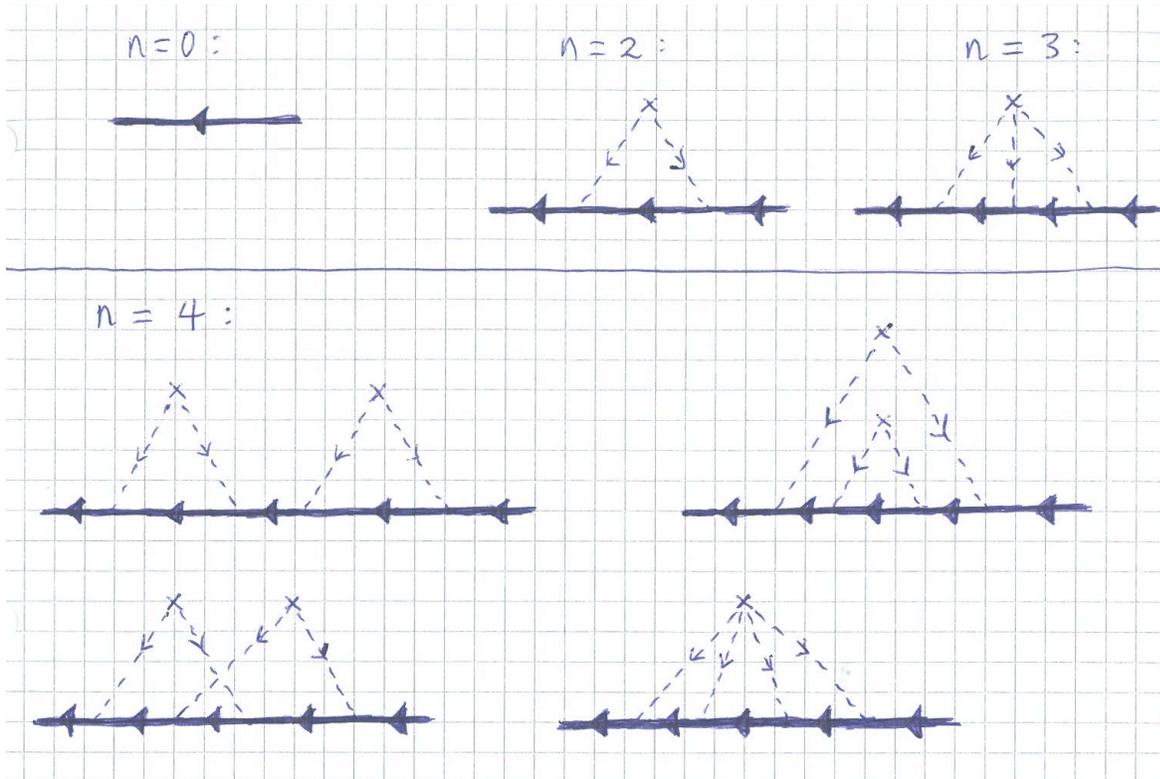
(c) We know from the Dyson equation that

$$\bar{\mathcal{G}} = \frac{1}{(\mathcal{G}^{(0)})^{-1} - \Sigma}. \quad (19)$$

This can be rewritten as

$$\bar{\mathcal{G}} = \frac{1}{(\mathcal{G}^{(0)})^{-1} - \Sigma^{(1)} - \Sigma'} = \frac{1}{(\tilde{\mathcal{G}}^{(0)})^{-1} - \tilde{\Sigma}}. \quad (20)$$

(d) The Feynman diagrams with up to $n = 4$ interaction (dashed) lines are shown in the figure below. The advantage of this perturbation expansion is that there are fewer diagrams than in the original perturbation expansion (there are no diagrams with trees).



(a) Both j^P and j^D are single-particle operators. We use that for a single-particle operator O , which in first quantization reads

$$O = \sum_{\vec{r}_i} o(\vec{r}_i)$$

the second quantized form, using the space-spin basis, is

$$O = \sum_{\sigma} \int d\vec{r}' \Psi_{\sigma}^{\dagger}(\vec{r}') o(\vec{r}') \Psi_{\sigma}(\vec{r}')$$

(Since the current operators depend on \vec{r} , we used a different symbol (here \vec{r}') for the integration variable)

For $\vec{j}^P(\vec{r})$ we have

$$o(\vec{r}) \rightarrow -\frac{ie}{2m} \left[\nabla_{\vec{r}'} \delta(\vec{r} - \vec{r}') + \delta(\vec{r} - \vec{r}') \nabla_{\vec{r}'} \right]$$

which gives the second-quantized expression

$$\vec{j}^P(\vec{r}) = -\frac{ie}{2m} \sum_{\sigma} \int d\vec{r}'$$

$$\cdot \Psi_{\sigma}^{\dagger}(\vec{r}') \left[\nabla_{\vec{r}'} \delta(\vec{r} - \vec{r}') + \delta(\vec{r} - \vec{r}') \nabla_{\vec{r}'} \right] \Psi_{\sigma}(\vec{r}')$$

In the second term, $\nabla_{\vec{r}'}$ acts only on $\Psi_{\sigma}(\vec{r}')$. Doing the integral for this term gives

$$-\frac{ie}{2m} \sum_{\sigma} \Psi_{\sigma}^{\dagger}(\vec{r}) \nabla_{\vec{r}} \Psi_{\sigma}(\vec{r})$$

In the first term, $\nabla_{\vec{r}'}$ acts on the product of $\delta(\vec{r} - \vec{r}')$ and $\Psi_{\sigma}(\vec{r}')$. We use integration by parts to instead have $\nabla_{\vec{r}'}$ acting on $\Psi_{\sigma}^{\dagger}(\vec{r}')$. This gives for the first term

$$\begin{aligned} \text{B.T.} &= -\left(-\frac{ie}{2m}\right) \sum_{\sigma} \int d\vec{r}' [\nabla_{\vec{r}'} \psi_{\sigma}^{\dagger}(\vec{r}')] \delta(\vec{r}-\vec{r}') \psi_{\sigma}(\vec{r}') \\ &= \text{B.T.} + \frac{ie}{2m} \sum_{\sigma} [\nabla_{\vec{r}} \psi_{\sigma}^{\dagger}(\vec{r})] \psi_{\sigma}(\vec{r}) \end{aligned}$$

where B.T. is the boundary terms from the integration by parts formula. To find this, we note that our system is a box (rectangular prism) with sides of length L_x, L_y, L_z . Because of the periodic boundary conditions,

$$\psi_{\sigma}(\vec{r} + \hat{e}_x L_x) = \psi_{\sigma}(\vec{r}) \quad (\text{and similarly for } y \text{ \& } z)$$

the contributions to the boundary term from opposite faces of the box will cancel, so the boundary term vanishes. Thus we get

$$j^p(\vec{r}) = -\frac{ie}{2m} \sum_{\sigma} (\psi_{\sigma}^{\dagger}(\vec{r}) \nabla \psi_{\sigma}(\vec{r}) - [\nabla \psi_{\sigma}^{\dagger}(\vec{r})] \psi_{\sigma}(\vec{r}))$$

For $\vec{j}^D(\vec{r})$ we have

$$o(\vec{r}') \rightarrow -\frac{e^2}{m} \vec{A}(\vec{r}') \delta(\vec{r}-\vec{r}')$$

which gives the second-quantized expression

$$\begin{aligned} \vec{j}^D(\vec{r}) &= -\frac{e^2}{2m} \sum_{\sigma} \int d\vec{r}' \psi_{\sigma}^{\dagger}(\vec{r}') \vec{A}(\vec{r}') \delta(\vec{r}-\vec{r}') \psi_{\sigma}(\vec{r}') \\ &= -\frac{e^2}{m} \vec{A}(\vec{r}) \sum_{\sigma} \psi_{\sigma}^{\dagger}(\vec{r}) \psi_{\sigma}(\vec{r}) \end{aligned}$$

(b) Here the result again follows by direct application of the formula used in (a), after identifying $o(\vec{r}') \rightarrow \frac{1}{2m} (-i\nabla_{\vec{r}'} - e\vec{A}(\vec{r}'))^2$.

(c) Extracting the part that is linear in \vec{A} gives

$$\frac{ie}{2m} \sum_{\sigma} \int d\vec{r} \psi_{\sigma}^{\dagger}(\vec{r}) [\nabla \cdot \vec{A}(\vec{r}) + \vec{A}(\vec{r}) \cdot \nabla] \psi_{\sigma}(\vec{r})$$

In the first term we use integration by parts to move ∇ to act on $\psi_{\sigma}^{\dagger}(\vec{r})$. The boundary term again vanishes due to the periodic boundary conditions. Thus we get

$$\begin{aligned} & \frac{ie}{2m} \sum_{\sigma} \int d\vec{r} \left\{ -[\nabla \psi_{\sigma}^{\dagger}(\vec{r})] \psi_{\sigma}(\vec{r}) + \psi_{\sigma}^{\dagger}(\vec{r}) \nabla \psi_{\sigma}(\vec{r}) \right\} \cdot \vec{A}(\vec{r}) \\ & = - \int d\vec{r} \vec{j}^p(\vec{r}) \cdot \vec{A}(\vec{r}) \end{aligned}$$