TFY4210, Quantum theory of many-particle systems, 2014: Tutorial 13

1. Alternative matrix representations.

In the lectures we discussed the standard ("Dirac-Pauli") representation of the γ-matrices in the Dirac equation. However, other, physically equivalent, representations are also possible. Consider a valid representation $\{\gamma^\mu\}$ (e.g. the Dirac-Pauli one) and define

$$\tilde{\gamma}^\mu = S\gamma^\mu S^\dagger \tag{1}$$

where S is a unitary 4×4 matrix. In this problem you will show that $\{\tilde{\gamma}^\mu\}$ is also a valid representation which gives physically equivalent results to the representation $\{\gamma^\mu\}$.

(a) Show that the matrices $\{\tilde{\gamma}^\mu\}$ satisfy the appropriate anticommutation relations, i.e. for all μ, ν,

$$\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2g^{\mu\nu}. \tag{2}$$

(b) Let Ψ be a solution of the Dirac equation expressed in terms of the $\{\gamma^\mu\}$ representation, i.e.

$$(i\hbar\gamma^\mu\partial_\mu - mc)\Psi = 0. \tag{3}$$

Show that the corresponding solution $\tilde{\Psi}$ of the Dirac equation expressed in terms of the $\{\tilde{\gamma}^\mu\}$ representation, i.e.

$$(i\hbar\tilde{\gamma}^\mu\partial_\mu - mc)\tilde{\Psi} = 0, \tag{4}$$

is given by $\tilde{\Psi} = S\Psi$.

(c) The probability density and probability current density affect physical quantities and must therefore be the same for the representations $\{\gamma^\mu\}$ and $\{\tilde{\gamma}^\mu\}$. Show that this is indeed the case, i.e. show that

$$\tilde{\rho} = \rho \quad \text{and} \quad \tilde{\boldsymbol{j}} = \boldsymbol{j}. \tag{5}$$

2. Eigenvalues of $\alpha_k\alpha_l$ ($k \neq l$).

Deduce the eigenvalues of the product $\alpha_k\alpha_l$ for $k \neq l$, which appears in the components of the spin operator for Dirac particles. (Solve the problem by making use of algebraic properties of the α matrices, avoid using an explicit matrix representation.)

3. Dirac equation for $m = 0$.

Dirac's linearization problem

$$\sqrt{p_x^2 + p_y^2 + p_z^2 + m^2c^2} = \alpha_1 p_x + \alpha_2 p_y + \alpha_3 p_z + \beta mc \tag{6}$$

simplifies if $m = 0$. Show that one can then use a 2-component wavefunction Ψ which satisfies

$$\frac{1}{c}\frac{\partial \Psi}{\partial t} = -\boldsymbol{\sigma} \cdot \nabla \Psi, \tag{7}$$

where the components of the vector $\boldsymbol{\sigma}$ are the Pauli matrices. (This equation is called the Weyl equation.)

4. Dirac equation in two spatial dimensions with an external magnetic field.

In this problem we consider the Dirac equation in two spatial dimensions (described by cartesian coordinates x and y). We set $c = 1$ and $\hbar = 1$ for simplicity.

(a) Show that a valid representation (to be used in the following) for the α and β matrices is

$$\beta = \sigma_3, \quad \alpha_1 = \sigma_1, \quad \alpha_2 = \sigma_2. \tag{8}$$

(b) An external magnetic field of magnitude B is applied in the z direction (i.e. perpendicular to the plane of motion): $\boldsymbol{B} = B\hat{z}$. In terms of electromagnetic potentials this can be represented as $\phi = 0$, $\boldsymbol{A} = (0, Bx, 0)$. Show that the Dirac equation for a particle of charge q in the presence of the magnetic field can be rewritten as

$$(i\sigma_3\partial_t - \sigma_2\partial_x + \sigma_1\partial_y - iqBx\sigma_1 - m)\Psi = 0. \tag{9}$$

(c) Explain why it is natural to use the ansatz (here p_y is a number)

$$\Psi(x, y, t) \quad = \quad e^{ip_y y - iEt}\begin{pmatrix} f(x) \\ g(x) \end{pmatrix}. \tag{10}$$

(d) Use the ansatz to derive an equation of the form $Q\begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = 0$, where Q is a 2×2 matrix. Simplify expressions by introducing the operators

$$\xi_\pm = -i\partial_x \mp i(p_y - qBx). \tag{11}$$

(e) Eliminate the function $g(x)$ from the problem to obtain an equation for $f(x)$ alone.

(f) Solve this equation, thus finding the solutions for both $f(x)$ and E (hint: harmonic oscillator). Finally, find $g(x)$.