Problem 1

(a) The electric field inside the conductor is zero.

(b) (i) $|\vec{r}| = R$: $V = V_0$ (a constant, i.e. independent of the direction of $\vec{r}$)

(ii) As $|\vec{r}| \to \infty$, the electric field $\vec{E}$ must approach the external field, i.e. $\vec{E} = E_0 \hat{z}$. Since $\vec{E} = -\nabla V$ this implies that $V = -E_0 z + V_\infty$ (here V_∞ is a constant, conventionally chosen to be 0).

(c) Since $z = r\cos\theta$, the boundary condition as $|\vec{r}| \to \infty$ can be written $V = -E_0 r P_1(\cos\theta) + V_\infty$. Comparing this with the expansion for $V(\vec{r})$ gives

$A_1 = -E_0$, $A_\ell = 0$ for $\ell \geq 2$ (and $A_0 = V_\infty$)
(we get no information about $\{B_\ell\}$).

Next we consider the boundary condition $V = V_0$ at $|\vec{r}| = R$, i.e. no θ-dependence. Since P_ℓ depends on θ for $\ell \geq 1$, the coefficient $A_\ell R^\ell + B_\ell / R^{\ell+1}$ must vanish for $\ell \geq 1$

$\Rightarrow B_\ell = -R^{2\ell+1} A_\ell$

$\ell \geq 2$: $B_\ell = -R^{2\ell+1} \cdot 0 = 0$

$\ell = 1$: $B_1 = -R^3 A_1 = R^3 E_0$

(and $\ell = 0$ gives $A_0 + B_0/R = V_0 \Rightarrow B_0 = R(V_0 - A_0)$ $= R(V_0 - V_\infty)$; however, these expressions for A_0 and B_0 are not necessary for what comes later).

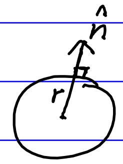
Thus terms with $\ell \geq 2$ do not contribute, leaving

$$V(\vec{r}) = \left(A_0 + \frac{B_0}{r}\right) + \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta$$

$$= A_0 + \frac{B_0}{r} + \left(-E_0 r + E_0 \frac{R^3}{r^2}\right)\cos\theta$$

(with $A_0 = V_\infty$, $B_0 = R(V_0 - V_\infty)$ "voluntary" results).

(d) We use the expression for σ in the formula set (Eq. (9) on p. 6). The "inside" term vanishes as it's proportional to $\vec{E}$ inside, giving

$$\sigma = -\epsilon_0 \left.\frac{\partial V_{\text{outside}}}{\partial r}\right|_{r=R} \qquad \left(\text{as } \frac{\partial}{\partial n} = \frac{\partial}{\partial r} : \right.$$  $)$

$$= -\epsilon_0 \frac{\partial}{\partial r}\left[A_0 + \frac{B_0}{r} - E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta \right]\Bigg|_{r=R}$$

$$= -\epsilon_0 \left[-\frac{B_0}{r^2} - E_0\left(1 - (-2)\frac{R^3}{r^3}\right)\cos\theta \right]\Bigg|_{r=R}$$

$$= \underline{\underline{\frac{\epsilon_0 B_0}{R^2} + 3\epsilon_0 E_0 \cos\theta}}$$

Total surface charge: $Q = \int \sigma\, dS = \underbrace{2\pi}_{\text{from } \varphi \text{ (azimuthal) integration}} R^2 \int_{-1}^{1} \sigma\, d(\cos\theta)$ $= 4\pi\epsilon_0 B_0$ (the term $\propto E_0$ is odd in $\cos\theta$ and thus its integral becomes 0). As $Q = 0$, it follows that $\underline{\underline{B_0 = 0}}$

(e) $p_z = \int d^3r\, z\, \rho(\vec{r})$. As the sphere is conducting, all charge resides on the surface $r = R$. The charge inside the solid angle $d\Omega$ is $d\varphi\, d(\cos\theta) \int_0^\infty dr\, r^2 \rho(\vec{r})$ $= d\varphi\, d(\cos\theta)\, R^2 \sigma \Rightarrow \rho(\vec{r}) = \sigma\, \delta(r-R)$. Thus (using $z = r\cos\theta$)

$$p_z = 2\pi \int_{-1}^{1} d(\cos\theta) \int_0^\infty dr\, r^2 \cdot (r\cos\theta)\left(3\epsilon_0 E_0 \cos\theta\right)\delta(r-R)$$

$$= 6\pi\epsilon_0 E_0 R^3 \underbrace{\int_{-1}^{1} \cos^2\theta\, d(\cos\theta)}_{\frac{1}{3}\left[1^3 - (-1)^3\right] = \frac{2}{3}} = \underline{\underline{4\pi R^3 \epsilon_0 E_0}}$$

Problem 2

(a) Maxwell equations (ME) in matter (see formula sheet)

$$\left.\begin{aligned} \nabla\cdot\vec{D} &= \rho_f \\ \nabla\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \\ \nabla\cdot\vec{B} &= 0 \\ \nabla\times\vec{H} &= \vec{J}_f + \frac{\partial\vec{D}}{\partial t} \end{aligned}\right\} \xrightarrow[\substack{\vec{D}=\varepsilon\vec{E} \\ \vec{B}=\mu\vec{H}}]{\rho_f = 0 = \vec{J}_f} \begin{aligned} \nabla\cdot\vec{E} &= 0 & (1) \\ \nabla\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} & (2) \\ \nabla\cdot\vec{B} &= 0 & (3) \\ \nabla\times\vec{B} &= \varepsilon\mu\frac{\partial\vec{E}}{\partial t} & (4) \end{aligned}$$

Also need (formula set): $\nabla\times(\nabla\times\vec{F}) = \nabla(\nabla\cdot\vec{F}) - \nabla^2\vec{F}$ (5)

$\nabla\times(2)$: LHS: $\nabla\times(\nabla\times\vec{E}) \overset{(5)}{=} \nabla\underbrace{(\nabla\cdot\vec{E})}_{=0\ (1)} - \nabla^2\vec{E} = \underline{-\nabla^2\vec{E}}$

RHS: $\nabla\times\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\nabla\times\vec{B} \underset{(4)}{=} \underline{-\varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}} \Rightarrow \underline{\underline{\nabla^2\vec{E} = \varepsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}}}$

$\nabla\times(4)$: LHS: $\nabla\times(\nabla\times\vec{B}) = \nabla\underbrace{(\nabla\cdot\vec{B})}_{=0\ (3)} - \nabla^2\vec{B} = -\nabla^2\vec{B}$

RHS: $\nabla\times\left(\varepsilon\mu\frac{\partial\vec{E}}{\partial t}\right) = \varepsilon\mu\frac{\partial}{\partial t}\nabla\times\vec{E} \underset{(2)}{=} -\varepsilon\mu\frac{\partial^2\vec{B}}{\partial t^2} \Rightarrow \underline{\underline{\nabla^2\vec{B} = \varepsilon\mu\frac{\partial^2\vec{B}}{\partial t^2}}}$

Thus both $\vec{E}$ and $\vec{B}$ satisfy the wave equation (see formula set) with $1/v^2 = \varepsilon\mu \Rightarrow \underline{\underline{v = 1/\sqrt{\varepsilon\mu}}}$

(b) Consider $\nabla\times\tilde{\vec{E}} = \nabla\times\left(\tilde{\vec{E}}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right)$

formula (7), p. 9

$$\overset{\downarrow}{=} e^{i(\vec{k}\cdot\vec{r}-\omega t)}\underbrace{\nabla\times\tilde{\vec{E}}_0}_{=0} - \tilde{\vec{E}}_0\times\underbrace{\nabla e^{i(\vec{k}\cdot\vec{r}-\omega t)}}_{i\vec{k}\,e^{\cdots}} = i\vec{k}\times\tilde{\vec{E}}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

By ME(2) this equals $-\frac{\partial\tilde{\vec{B}}}{\partial t} = -(-i\omega)\tilde{\vec{B}}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$\Rightarrow \vec{k}\times\tilde{\vec{E}}_0 = \omega\tilde{\vec{B}}_0$

Writing $\tilde{\vec{E}}_0 = \vec{E}_0\, e^{i\delta_E}$, $\tilde{\vec{B}}_0 = \vec{B}_0\, e^{i\delta_B}$ with $\vec{E}_0, \vec{B}_0$ real

$\Rightarrow \underbrace{(\vec{k}\times\vec{E}_0)}_{\text{real}}\, e^{i\delta_E} = \underbrace{\omega\vec{B}_0}_{\text{real}}\, e^{i\delta_B}$

Solved if $e^{i\delta_E} = e^{i\delta_B}$ (i.e. $\vec{E}$ and $\vec{B}$ are in phase)
and $\vec{k} \times \vec{E}_0 = \omega \vec{B}_0$ (so $\vec{B}_0$ is $\perp$ to $\vec{k}$ and $\vec{E}_0$)

Also, $\nabla \cdot \tilde{\vec{E}} = 0 \Rightarrow \vec{k} \cdot \vec{E}_0 = 0$ (*)

$\Rightarrow \vec{k}, \vec{E}_0, \vec{B}_0$ orthogonal & right-handed system
$\Rightarrow$ $\vec{E}_0, \vec{B}_0, \vec{k}$ orthogonal & right-handed (by cyclic perm.)

Finally, $|\vec{k} \times \vec{E}_0| \overset{(*)}{=} k|\vec{E}_0| = \omega |\vec{B}_0| \Rightarrow |\vec{B}_0| = \frac{k}{\omega}|\vec{E}_0| = \frac{|\vec{E}_0|}{v}$

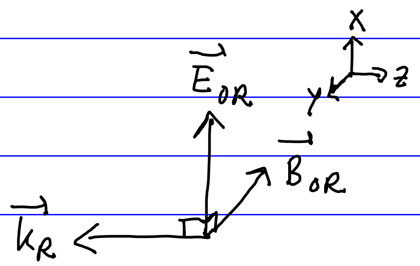
$\Rightarrow |\vec{B}| = |\vec{E}|/v$

used formula set, replaced $\mu_0 \to \mu$ for vacuum $\to$ matter

Intensity : $I = |\langle \vec{S} \rangle|$, $\vec{S} = \frac{1}{\mu}(\vec{E} \times \vec{B})$

$|\vec{E}| \propto E_0$, $|\vec{B}| \propto E_0$ (since $B_0 = E_0/v$) $\Rightarrow I \propto E_0^2$

(c) Reflected wave:



$$\tilde{\vec{E}}_R(z,t) = \tilde{E}_{OR}\, e^{i(-k_1 z - \omega t)}\, \hat{x}$$

$$\vec{B}_R(z,t) = -\frac{\tilde{E}_{OR}}{v_1}\, e^{i(-k_1 z - \omega t)}\, \hat{y}$$

Transmitted wave (same form as incident, with $I \to T$, $1 \to 2$)

$$\tilde{\vec{E}}_T(z,t) = \tilde{E}_{OT}\, e^{i(k_2 z - \omega t)}\, \hat{x}$$

$$\vec{B}_T(z,t) = \frac{\tilde{E}_{OT}}{v_2}\, e^{i(k_2 z - \omega t)}\, \hat{y}$$

(d) $R = \frac{I_R}{I_I}$. Since $I_R \propto E_{OR}^2$, $I_I \propto E_{OI}^2$, and the proportionality constants are identical (since both the incident & reflected wave propagate in the same medium (1)), we get

$$R = \left(\frac{E_{OR}}{E_{OI}}\right)^2$$

Boundary conditions (see formula set):

$\vec{D}_1^{\perp} = \vec{D}_2^{\perp}$, $\vec{B}_1^{\perp} = \vec{B}_2^{\perp}$, $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$, $\vec{H}_1^{\parallel} = \vec{H}_2^{\parallel}$

As all $\perp$-fields vanish here, this gives $\underline{\vec{E}_1 = \vec{E}_2, \ \frac{\vec{B}_1}{\mu_1} = \frac{\vec{B}_2}{\mu_2}}$

Also, we have $\vec{E}_1 = \vec{E}_I + \vec{E}_R$, $\vec{B}_1 = \vec{B}_I + \vec{B}_R$, $\vec{E}_2 = \vec{E}_T$, $\vec{B}_2 = \vec{B}_T$. Applying the boundary conditions (at $z=0$, for all t) then gives

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T} \qquad (1')$$

$$\frac{1}{\mu_1}\left(\frac{\tilde{E}_{0I}}{v_1} - \frac{\tilde{E}_{0R}}{v_1}\right) = \frac{1}{\mu_2}\frac{\tilde{E}_{0T}}{v_2} \qquad (2')$$

Put (1') into (2') $\Rightarrow$ $\frac{1}{\mu_1 v_1}(\tilde{E}_{0I} - \tilde{E}_{0R}) = \frac{1}{\mu_2 v_2}(\tilde{E}_{0I} + \tilde{E}_{0R})$

Define $x \equiv \tilde{E}_{0R}/\tilde{E}_{0I}$ $\Rightarrow$ $\frac{1}{\mu_1 v_1}(1-x) = \frac{1}{\mu_2 v_2}(1+x)$

$$\Rightarrow x\left(\frac{1}{\mu_1 v_1} + \frac{1}{\mu_2 v_2}\right) = \frac{1}{\mu_1 v_1} - \frac{1}{\mu_2 v_2}$$

$$\Rightarrow x(\mu_2 v_2 + \mu_1 v_1) = \mu_2 v_2 - \mu_1 v_1$$

$$\Rightarrow x = \frac{\mu_2 v_2 - \mu_1 v_1}{\mu_2 v_2 + \mu_1 v_1} \quad \text{(real, since } \mu_i, v_i \text{ real)}$$

$$\Rightarrow R = x^2 = \left(\frac{\mu_2 v_2 - \mu_1 v_1}{\mu_2 v_2 + \mu_1 v_1}\right)^2 \quad \left(\text{now use } v_i = \frac{1}{\sqrt{\epsilon_i \mu_i}}\right)$$

$$= \underline{\underline{\left(\frac{\sqrt{\frac{\mu_1}{\epsilon_1}} - \sqrt{\frac{\mu_2}{\epsilon_2}}}{\sqrt{\frac{\mu_1}{\epsilon_1}} + \sqrt{\frac{\mu_2}{\epsilon_2}}}\right)^2}}$$

Problem 3

(a) Retarded time: $t_r = t - \frac{|\vec{r}-\vec{r}'|}{c}$

Interpretation: the contribution to the potential at $\vec{r}$ at time t from a source at $\vec{r}'$ occurs through that source's state at time $t_r < t$, with $t - t_r$ being the time the "signal" takes to travel from $\vec{r}'$ to $\vec{r}$ at the speed of light.

(b) The wire is electrically neutral, i.e. there is no net charge density ($\rho(\vec{r},t')=0$) which immediately gives $V(\vec{r},t)=0$

(c) As the current density $\vec{J}$ is along the z axis, so is $\vec{A}$, i.e. $\vec{A}(\vec{r},t) = A_z(\vec{r},t)\hat{z}$ with

$$A_z(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{J(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|} \qquad [\vec{r}' \equiv (x',y',z')]$$

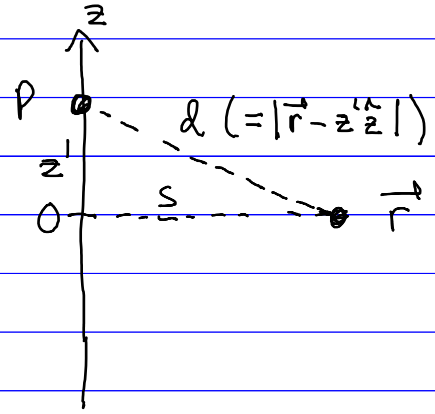
Since J is zero outside the wire, and as a current density should satisfy $\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \, J = I$, we can write

$$\vec{J}(\vec{r}',t_r) = I(t_r)\,\delta(x')\,\delta(y')$$

$$\Rightarrow A_z(\vec{r},t) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} dz' \, \frac{I(t_r)}{|\vec{r}-z'\hat{z}|}$$

Because of the symmetry of the problem, $A_z(\vec{r},t)$ can not depend on φ or z.

So we can evaluate the integral for $z=0$ (and an arbitrary φ)

$\Rightarrow$ 

A point P on the wire at $\vec{r}\,' = z'\hat{z}$ is at distance $d = \sqrt{z'^2 + s^2}$ from $\vec{r}$. For P to contribute to $\vec{A}$ at $(\vec{r}, t)$ requires $d \le ct$

$$\Rightarrow \quad |z'|^2 + s^2 \le (ct)^2 \Rightarrow |z'| \le \sqrt{(ct)^2 - s^2} \; (\equiv z'_{max})$$

For $s > ct$, no z' satisfy this $\Rightarrow A_z(\vec{r}, t) = 0$.
For $s < ct$, only z' between $\pm z'_{max}$ contribute, giving

$$A_z(\vec{r}, t) = \frac{\mu_0 I_0}{4\pi} \int_{-z'_{max}}^{z'_{max}} \frac{dz'}{\sqrt{z'^2 + s^2}} = \frac{\mu_0 I_0}{2\pi} \int_0^{z'_{max}} \frac{dz'}{\sqrt{z'^2 + s^2}}$$

Using Rottmann, the integral is

$$\ln\left(\sqrt{z'^2 + s^2} + z'\right)\Big|_0^{z'_{max}}$$

$$= \ln\left(\sqrt{z'^2_{max} + s^2} + z'_{max}\right) - \ln|s| \quad (s \ge 0 \text{ by def since it is a radial coord})$$

$$= \ln\left(ct + \sqrt{(ct)^2 - s^2}\right) - \ln s$$

$$= \ln\left(\frac{ct + \sqrt{(ct)^2 - s^2}}{s}\right)$$

$$\Rightarrow \quad A_z(\vec{r}, t) = \begin{cases} 0 & t < s/c \\ \dfrac{\mu_0 I_0}{2\pi} \ln\left(\dfrac{ct + \sqrt{(ct)^2 - s^2}}{s}\right) & t > s/c \end{cases}$$

(d) $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} = -\frac{\partial \vec{A}}{\partial t}$; $\vec{B} = \nabla \times \vec{A}$

For $t < s/c$, $\vec{E}$ and $\vec{B}$ are both zero.
For $t > s/c$, we find

$$\vec{E}(\bar{r},t) = -\hat{z}\,\frac{\mu_0 I_0}{2\pi}\,\frac{\partial}{\partial t}\ln\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right)$$

$$= -\hat{z}\,\frac{\mu_0 I_0}{2\pi}\cdot\frac{\cancel{s}}{ct+\sqrt{(ct)^2-s^2}}\cdot\frac{1}{\cancel{s}}\cdot\left(c+\frac{\cancel{2}ct\cdot c}{\cancel{2}\sqrt{(ct)^2-s^2}}\right)$$

$$= -\hat{z}\,\frac{\mu_0 I_0}{2\pi}\,\frac{1}{\cancel{ct+\sqrt{(ct)^2-s^2}}}\cdot\frac{c\left(\cancel{\sqrt{(ct)^2-s^2}+ct}\right)}{\sqrt{(ct)^2-s^2}}$$

$$= \underline{\underline{-\frac{\mu_0 I_0 c}{2\pi\sqrt{(ct)^2-s^2}}\,\hat{z}}}$$

Next, looking at the expression for the curl in cylinder coords (see formula set), and using that A_z only depends on s, not z or φ, we get

$$\vec{B} = \nabla\times\vec{A} = -\frac{\partial A_z}{\partial s}\,\hat{\varphi} = -\hat{\varphi}\,\frac{\mu_0 I_0}{2\pi}\,\frac{\partial}{\partial s}\ln\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right)$$

$$= -\hat{\varphi}\,\frac{\mu_0 I_0}{2\pi}\,\frac{s}{ct+\sqrt{(ct)^2-s^2}}\left[\frac{1}{s}\,\frac{(-\cancel{2}s)}{\cancel{2}\sqrt{(ct)^2-s^2}}+\left(-\frac{1}{s^2}\right)\left(ct+\sqrt{(ct)^2-s^2}\right)\right]$$

$$= \hat{\varphi}\,\frac{\mu_0 I_0}{2\pi}\,\frac{\cancel{s}}{ct+\sqrt{(ct)^2-s^2}}\cdot\frac{\cancel{s^2}+ct\sqrt{(ct)^2-s^2}+(ct)^2-\cancel{s^2}}{s^{\cancel{2}}\sqrt{(ct)^2-s^2}}$$

$$= \underline{\underline{\frac{\mu_0 I_0}{2\pi s}\,\frac{ct}{\sqrt{(ct)^2-s^2}}\,\hat{\varphi}}}$$

(e) As $t\to\infty$, $\vec{E}\to 0$ and $\vec{B}\to\frac{\mu_0 I_0}{2\pi s}\hat{\varphi}$, which are the expressions for $\vec{E}$ and $\vec{B}$ for the <u>magnetostatic</u> problem of a wire carrying a constant current I_0, exactly as expected in this limit when the current has been on "forever".