# Solution Exercise 1

## Problem 2

a) $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

Looking at the i'th component of the resulting vector and get

$$\varepsilon_{ijk} A_j (\vec{B} \times \vec{C})_k$$

$$= \varepsilon_{ijk} A_j \varepsilon_{klm} B_l C_m$$

$$= \varepsilon_{ijk} \varepsilon_{klm} A_j B_l C_m$$

$$= - \varepsilon_{kji} \varepsilon_{klm} A_j B_l C_m$$

$$= -(\delta_{jl} \delta_{im} - \delta_{jm} \delta_{il}) A_j B_l C_m$$

$$= - A_j B_j C_i + A_j B_i C_j$$

$$= B_i A_j C_j - C_i A_j B_j$$

$$= B_i (\vec{A} \cdot \vec{C}) - C_i (\vec{A} \cdot \vec{B})$$

Since this holds for any $i = 1,2,3$ we have proved the given relation.

b) $\nabla \cdot (\nabla \times \vec{A}) = 0$

$$\partial_i (\nabla \times \vec{A})_i = \partial_i \varepsilon_{ijk} \partial_j A_k$$

$$= \varepsilon_{ijk} \partial_i \partial_j A_k$$

$$= 0$$

This follows from the fact that $\varepsilon_{ijk}$ is anti-symmetric in $i$ and $j$, while $\partial_i \partial_j$ is symmetric in the same indices. This is in fact a general result that the product of a symmetric and an anti-symmetric tensor equals zero.

In our case the proof goes like this.

$$\varepsilon_{ijk} \partial_i \partial_j = - \varepsilon_{jik} \partial_i \partial_j$$

$$= - \varepsilon_{ijk} \partial_j \partial_i \quad \text{rename } i \leftrightarrow j$$

$$= - \varepsilon_{ijk} \partial_i \partial_j$$

$$\Rightarrow \varepsilon_{ijk} \partial_i \partial_j = 0$$

c) $\nabla \times (\nabla \psi) = 0$

$$\varepsilon_{ijk}\, \partial_j \partial_k \psi = 0$$

Again we have a product of a sym. and anti-sym. tensor that is zero for the same reason as under point b)

d) $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

This resembles # a), but make sure that $\partial_i$ operates on something so that the order of the operators may be important.

$$\varepsilon_{ijk}\, \partial_j\, \varepsilon_{klm}\, \partial_l A_m$$

$$= \varepsilon_{ijk}\, \varepsilon_{klm}\, \partial_j \partial_l A_m$$

$$= -(\delta_{jl}\delta_{im} - \delta_{jm}\delta_{il})\, \partial_j \partial_l A_m$$

$$= -\partial_j \partial_j A_i + \underbrace{\partial_j \partial_i A_j}_{\partial_i(\partial_j A_j)}$$

$$= \partial_i(\nabla \cdot \vec{A}) - \nabla^2 A_i$$

e) $\nabla \times \left(\frac{\vec{A}}{g}\right) = \frac{g(\nabla \times \vec{A}) + \vec{A} \times (\nabla g)}{g^2}$

$$\varepsilon_{ijk} \partial_j \left(\frac{A_k}{g}\right)$$

$$= \varepsilon_{ijk} \frac{g \cdot \partial_j A_k - A_k \cdot \partial_j g}{g^2}$$

$$= \frac{g(\nabla \times \vec{A})_i - \varepsilon_{ijk} (\partial_j g) A_k}{g^2}$$

$$= \frac{g(\nabla \times \vec{A})_i - ((\nabla g) \times \vec{A})_i}{g^2}$$

$$= \frac{g(\nabla \times \vec{A})_i + (\vec{A} \times \nabla g)_i}{g^2} \quad ; \quad \vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$