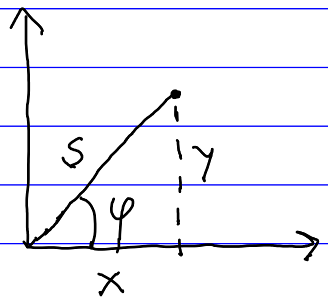
Problem 1

$$Df = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

Rewrite this to the cylindrical coord. system with coordinates (s, φ, z). Since the coordinate z is still used, the term $\frac{\partial f}{\partial z}\hat{z}$ is unchanged. Thus it remains to consider

$$\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y}$$

Need to rewrite $\hat{x}$ and $\hat{y}$ in terms of $\hat{s}$ and $\hat{\varphi}$, and rewrite $\partial f/\partial x$ and $\partial f/\partial y$ in terms of $\partial f/\partial r$ and $\partial f/\partial \varphi$.



Use the chain rule:

$$\frac{\partial f}{\partial x} = \frac{\partial s}{\partial x}\frac{\partial f}{\partial s} + \frac{\partial \varphi}{\partial x}\frac{\partial f}{\partial \varphi}$$

$$\frac{\partial f}{\partial y} = \frac{\partial s}{\partial y}\frac{\partial f}{\partial s} + \frac{\partial \varphi}{\partial y}\frac{\partial f}{\partial \varphi}$$

$$s = \sqrt{x^2 + y^2}$$

$$\Rightarrow \frac{\partial s}{\partial x} = \frac{2x}{\sqrt{x^2+y^2}} = \frac{x}{s} = \frac{s\cos\varphi}{s} = \underline{\cos\varphi}$$

$$\frac{\partial s}{\partial y} = \frac{2y}{\sqrt{x^2+y^2}} = \underline{\sin\varphi}$$

$$\tan\varphi = \frac{y}{x} \Rightarrow \varphi = \arctan\left(\frac{y}{x}\right)$$

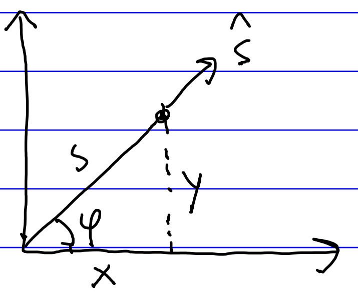
$$\Rightarrow \frac{\partial \varphi}{\partial x} = \frac{1}{1+(y/x)^2}\cdot y \cdot\left(-\frac{1}{x^2}\right) = -\frac{y}{x^2+y^2} = \underline{-\frac{1}{s}\sin\varphi}$$

$$\frac{\partial\varphi}{\partial y} = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2} = \underline{\frac{1}{s}\cos\varphi}$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = \cos\varphi\,\frac{\partial f}{\partial s} - \frac{1}{s}\sin\varphi\,\frac{\partial f}{\partial\varphi}$$

$$\frac{\partial f}{\partial y} = \sin\varphi\,\frac{\partial f}{\partial s} + \frac{1}{s}\cos\varphi\,\frac{\partial f}{\partial\varphi}$$

Next, relate $\hat{x}$ and $\hat{y}$ to $\hat{s}$ and $\hat{\varphi}$:



$$\hat{s} = \frac{x\hat{x}+y\hat{y}}{|x\hat{x}+y\hat{y}|} = \frac{x\hat{x}+y\hat{y}}{\sqrt{x^2+y^2}}$$

$$= \frac{s\cos\varphi\,\hat{x} + s\sin\varphi\,\hat{y}}{s}$$

$$= \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}$$

Write $\hat{\varphi} = a\hat{x} + b\hat{y}$

$$\hat{\varphi}\cdot\hat{\varphi} = 1 \Rightarrow a^2+b^2 = 1$$

$$\hat{\varphi}\cdot\hat{s} = 0 \Rightarrow a\cos\varphi + b\sin\varphi = 0$$

$$\Rightarrow b = -a\,\frac{\cos\varphi}{\sin\varphi}$$

$$\Rightarrow a^2+b^2 = a^2\left(1+\frac{\cos^2\varphi}{\sin^2\varphi}\right) = 1$$

$$\Rightarrow a^2 = \frac{1}{1+\cos^2\varphi/\sin^2\varphi} = \frac{\sin^2\varphi}{\sin^2\varphi+\cos^2\varphi} = \sin^2\varphi$$

$$\Rightarrow a = \pm\sin\varphi$$

$$\Rightarrow \hat{\varphi} = \pm\left(\sin\varphi\,\hat{x} - \cos\varphi\,\hat{y}\right)$$

To get a right-handed system, we should have $\hat{\varphi} = +\hat{y}$ when $\varphi = 0 \Rightarrow$ pick sign $= -1$

Summarized:

$$\hat{s} = \cos\varphi\, \hat{x} + \sin\varphi\, \hat{y}$$
$$\hat{\varphi} = -\sin\varphi\, \hat{x} + \cos\varphi\, \hat{y}$$

$$\Rightarrow \quad \cos\varphi\, \hat{s} - \sin\varphi\, \hat{\varphi} = \hat{x}$$
$$\sin\varphi\, \hat{s} + \cos\varphi\, \hat{\varphi} = \hat{y}$$

Now we are ready to do the main calculation:

$$\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y}$$

$$= \left(\cos\varphi \frac{\partial f}{\partial s} - \frac{1}{s}\sin\varphi \frac{\partial f}{\partial \varphi}\right)\left(\cos\varphi\, \hat{s} - \sin\varphi\, \hat{\varphi}\right)$$

$$+ \left(\sin\varphi \frac{\partial f}{\partial s} + \frac{1}{s}\cos\varphi \frac{\partial f}{\partial \varphi}\right)\left(\sin\varphi\, \hat{s} + \cos\varphi\, \hat{\varphi}\right)$$

$$= \hat{s}\left[\cos^2\varphi \frac{\partial f}{\partial s} - \frac{1}{s}\sin\varphi\cos\varphi \frac{\partial f}{\partial \varphi} + \sin^2\varphi \frac{\partial f}{\partial s} + \frac{1}{s}\cos\varphi\sin\varphi \frac{\partial f}{\partial \varphi}\right]$$

$$+ \hat{\varphi}\left[-\cos\varphi\sin\varphi \frac{\partial f}{\partial s} + \frac{1}{s}\sin^2\varphi \frac{\partial f}{\partial \varphi} + \sin\varphi\cos\varphi \frac{\partial f}{\partial s} + \frac{1}{s}\cos^2\varphi \frac{\partial f}{\partial \varphi}\right]$$

$$= \underline{\underline{\frac{\partial f}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial f}{\partial \varphi}\hat{\varphi}}} \qquad \text{Q.E.D.}$$

Problem 2

(a) $\epsilon_{ijk}\,\epsilon_{ilm} = \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}$

(i) $j = l = 1,\quad k = m = 3$

LHS : $\epsilon_{i13}\,\epsilon_{i13} = \epsilon_{113}\,\epsilon_{113} + \underbrace{\epsilon_{213}\,\epsilon_{213}}_{(\epsilon_{213})^2} + \epsilon_{313}\,\epsilon_{313}$

$= 0 + 1 + 0 = \underline{\underline{1}}$

RHS : $\delta_{11}\,\delta_{33} - \delta_{13}\,\delta_{31} = 1 \cdot 1 - 0 \cdot 0 = \underline{\underline{1}}$

(ii) $j = m = 1,\quad k = l = 2$

LHS : $\epsilon_{i12}\,\epsilon_{i21} = \epsilon_{112}\,\epsilon_{121} + \epsilon_{212}\,\epsilon_{221} + \underbrace{\epsilon_{312}\,\epsilon_{321}}_{-\epsilon_{321}^2}$

$= 0 + 0 + (-1) = \underline{\underline{-1}}$

RHS : $\delta_{12}\,\delta_{21} - \delta_{11}\,\delta_{22} = 0 \cdot 0 - 1 \cdot 1 = \underline{\underline{-1}}$

(iii) $j = l = 1,\quad k = 2,\quad m = 3$:

LHS: $\epsilon_{i12}\,\epsilon_{i13} = \epsilon_{112}\,\epsilon_{113} + \epsilon_{212}\,\epsilon_{213} + \epsilon_{312}\,\epsilon_{313}$

$= 0 + 0 + 0 = \underline{\underline{0}}$

RHS: $\delta_{11}\,\delta_{23} - \delta_{13}\,\delta_{21} = 1 \cdot 0 - 0 \cdot 0 = \underline{\underline{0}}$

(b)

(i) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = \underline{\underline{3}}$

(ii) $\delta_{ij}\, \epsilon_{ijk} = \epsilon_{iik} \; (= \epsilon_{11k} + \epsilon_{22k} + \epsilon_{33k}) = \underline{\underline{0}}$

(iii) $\epsilon_{ijk}\, \epsilon_{ljk} = \epsilon_{kij}\, \epsilon_{klj}$

$= \delta_{il}\, \delta_{jj} - \delta_{ij}\, \delta_{jl} = 3\delta_{il} - \delta_{il} = \underline{\underline{2\delta_{il}}}$

(iv) $\epsilon_{ijk}\, \epsilon_{ijk} = \delta_{jj}\, \delta_{kk} - \delta_{jk}\, \delta_{kj}$

$= 3.3 - \delta_{kk} = 9 - 3 = \underline{\underline{6}}$

Solution Exercise 1

Problem 2

a) $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

Looking at the i'th component of the resulting vector and get

$$\varepsilon_{ijk} A_j (\vec{B} \times \vec{C})_k$$

$$= \varepsilon_{ijk} A_j \varepsilon_{klm} B_l C_m$$

$$= \varepsilon_{ijk} \varepsilon_{klm} A_j B_l C_m$$

$$= - \varepsilon_{kji} \varepsilon_{klm} A_j B_l C_m$$

$$= -(\delta_{jl}\delta_{im} - \delta_{jm}\delta_{il}) A_j B_l C_m$$

$$= - A_j B_j C_i + A_j B_i C_j$$

$$= B_i A_j C_j - C_i A_j B_j$$

$$= B_i (\vec{A}\cdot\vec{C}) - C_i (\vec{A}\cdot\vec{B})$$

Since this holds for any $i = 1,2,3$ we have proved the given relation.

b) $\nabla \cdot (\nabla \times \vec{A}) = 0$

$$\partial_i (\nabla \times \vec{A})_i = \partial_i \varepsilon_{ijk} \partial_j A_k$$

$$= \varepsilon_{ijk} \partial_i \partial_j A_k$$

$$= 0$$

This follows from the fact that ε_{ijk} is anti-symmetric in i and j, while $\partial_i \partial_j$ is symmetric in the same indices. This is in fact a general result that the product of a symmetric and an anti-symmetric tensor equals zero.

In our case the proof goes like this.

$$\varepsilon_{ijk} \partial_i \partial_j = - \varepsilon_{jik} \partial_i \partial_j$$

$$= - \varepsilon_{ijk} \partial_j \partial_i \quad \text{rename } i \leftrightarrow j$$

$$= - \varepsilon_{ijk} \partial_i \partial_j$$

$$\Rightarrow \varepsilon_{ijk} \partial_i \partial_j = 0$$

c) $\nabla \times (\nabla \psi) = 0$

$$\varepsilon_{ijk} \partial_j \partial_k \psi = 0$$

Again we have a product of a sym. and anti-sym. tensor that is zero for the same reason as under point b)

d) $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

This resembles # a), but make sure that ∂_i operates on something so that the order of the operators may be important.

$$\varepsilon_{ijk} \partial_j \varepsilon_{klm} \partial_l A_m$$

$$= \varepsilon_{ijk} \varepsilon_{klm} \partial_j \partial_l A_m$$

$$= -(\delta_{jl}\delta_{im} - \delta_{jm}\delta_{il}) \partial_j \partial_l A_m$$

$$= -\partial_j \partial_j A_i + \underbrace{\partial_j \partial_i A_j}_{\partial_i(\partial_j A_j)}$$

$$= \partial_i (\nabla \cdot \vec{A}) - \nabla^2 A_i$$

(e) $\left[\nabla \times (f\vec{A})\right]_i = \epsilon_{ijk}\,\partial_j (f A_k)$

$= \epsilon_{ijk}\left[f\,\partial_j A_k + A_k\,\partial_j f \right]$

$= f\,\epsilon_{ijk}\,\partial_j A_k - \epsilon_{ikj}\,A_k\,\partial_j f$

$= f\,(\nabla \times \vec{A})_i - (\vec{A} \times (\nabla f))_i$