Solution to tutorial 2

Problem 1

(a) $$\underline{\underline{\rho(\vec{r},t) = \sum_{\alpha=1}^{N} q_\alpha \, \delta(\vec{r}-\vec{r}_\alpha(t))}}$$

(b) Continuity eq.: $\frac{\partial \rho}{\partial t} = -\nabla\cdot\vec{j}$

$$\frac{\partial \rho}{\partial t} = \sum_\alpha q_\alpha \frac{\partial}{\partial t}\delta(\vec{r}-\vec{r}_\alpha)$$

$$= \sum_\alpha q_\alpha \underbrace{\frac{d\vec{r}_\alpha}{dt}}_{\vec{v}_\alpha} \cdot \underbrace{\nabla_{\vec{r}_\alpha}\delta(\vec{r}-\vec{r}_\alpha)}_{= -\nabla\delta(\vec{r}-\vec{r}_\alpha)} \quad \left(\nabla \equiv \nabla_{\vec{r}} \text{ as always}\right)$$

since $\nabla\cdot\vec{v}_\alpha = 0$

$$\stackrel{\downarrow}{=} -\nabla\cdot\left(\sum_\alpha q_\alpha \vec{v}_\alpha \, \delta(\vec{r}-\vec{r}_\alpha)\right)$$

$$\Rightarrow \quad \underline{\underline{\vec{j}(\vec{r},t) = \sum_{\alpha=1}^{N} q_\alpha \vec{v}_\alpha(t)\, \delta(\vec{r}-\vec{r}_\alpha(t))}}$$

(the continuity eq. would allow an additional term $\vec{w}$ satisfying $\nabla\cdot\vec{w}=0$, but requiring $\vec{j}=0$ if all $\vec{v}_\alpha = 0$ gives $\vec{w}=0$)

Problem 2

$$\rho(r,\theta,\varphi) = \sigma(\theta,\varphi)\,\delta(r-R)$$

Properties:

- As σ has dimensions of charge per area and $\delta(r-R)$ has dimensions of inverse length, ρ has dimensions of charge per (area · length) = charge per volume

- $\rho(r \neq R) = 0$ since $\delta(r-R) = 0$ for $r \neq R$

- volume element in spherical coords:
$d^3r = d\varphi \sin\theta \, d\theta \, r^2 \, dr$

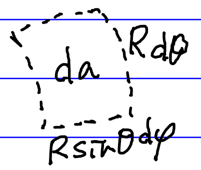
Charge in volume element $d^3r = \rho \, d^3r$

$= \sigma(\theta,\varphi) \, \delta(r-R) \, d\varphi \sin\theta \, d\theta \, r^2 \, dr$

Integrating this from $r = R^-$ to R^+ gives the charge

$$\sigma(\theta,\varphi) \, d\varphi \sin\theta \, d\theta \underbrace{\int_{R^-}^{R^+} dr \, r^2 \, \delta(r-R)}_{= R^2}$$

$$\simeq \sigma(\theta,\varphi) \, R \sin\theta \, d\varphi \, R \, d\theta = \underline{\underline{\sigma(\theta,\varphi) \, da}}$$



Problem 3

Since ∇ acts on $\vec{r}$ we can move it past the purely $\vec{r}\,'$-dependent quantities, giving

$$\nabla \times \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \rho(\vec{r}\,') \, \nabla \times \frac{\hat{R}}{R^2}$$

Let us first evaluate this for $\vec{r}\,' = 0$ and later shift to arbitrary $\vec{r}\,'$ (as done in the lecture notes). Thus consider

$$\nabla \times \frac{\hat{r}}{r^2} .$$

Using the identity $\nabla \times (f\vec{A}) = f\nabla\times\vec{A} - \vec{A}\times\nabla f$ (*)

$$\nabla \times \frac{\hat{r}}{r^2} = \frac{1}{r^2}\nabla\times\hat{r} - \hat{r}\times\left(\nabla\frac{1}{r^2}\right)$$

$$\hat{r} = v_r\,\hat{r} + v_\theta\,\hat{\theta} + v_\varphi\,\hat{\varphi} \quad \text{with } v_r = 1,\ v_\theta = v_\varphi = 0$$

$$\Rightarrow \nabla\times\hat{r} = 0$$

$$\nabla\frac{1}{r^2} = \hat{r}\,\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right) = -\frac{2}{r^3}\,\hat{r} \quad \Rightarrow \quad \hat{r}\times\nabla\frac{1}{r^2} = 0$$

where we used expressions for the curl and gradient in spherical coords

$$\Rightarrow \nabla\times\frac{\hat{r}}{r^2} = 0$$

Doing the shift $\vec{r}\to\vec{r}-\vec{r}\,'$ then also gives

$$\nabla\times\frac{\hat{R}}{R^2} = 0 \quad \Rightarrow \quad \nabla\times\vec{E} = 0$$

As an alternative, here is a derivation using the cartesian coordinate system:

$$\nabla\times\frac{\hat{R}}{R^2} = \nabla\times\frac{\vec{R}}{R^3} \overset{(*)}{=} \frac{1}{R^3}\nabla\times\vec{R} - \vec{R}\times\nabla\frac{1}{R^3}$$

$$\vec{R} = \vec{r} - \vec{r}\,' \Rightarrow R_k = x_k - x_k'$$

$$(\nabla\times\vec{R})_i = \epsilon_{ijk}\,\partial_j(x_k - x_k') = \epsilon_{ijk}\,\delta_{jk} = \epsilon_{ijj} = 0$$

$$\Rightarrow \nabla\times\vec{R} = 0$$

$$\partial_i \frac{1}{R^3} = \left(\frac{\partial}{\partial R}\frac{1}{R^3}\right)\partial_i R$$

$$= -\frac{3}{R^4}\,\partial_i\left[(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2\right]^{1/2}$$

$$= -\frac{3}{R^4}\cdot\frac{2(x_i-x_i')\cdot 1}{2R} = -\frac{3(x_i-x_i')}{R^5}$$

$$\Rightarrow \nabla\frac{1}{R^3} = \hat{e}_i\,\partial_i\frac{1}{R^3} = -\frac{3}{R^5}\,\hat{e}_i(x_i-x_i') = -\frac{3\vec{R}}{R^5}$$

$$\Rightarrow \vec{R}\times\nabla\frac{1}{R^3} = -\frac{3}{R^5}(\vec{R}\times\vec{R}) = 0$$

$$\Rightarrow \nabla\times\frac{\hat{R}}{R^2} = 0 \quad\Rightarrow\quad \nabla\times\vec{E} = 0$$

Problem 4

See Griffiths

Problem 5

(a) The potential is, for $|\boldsymbol{r}| \geq R$,

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\boldsymbol{r}-a\hat{\boldsymbol{z}}|} + \frac{q'}{|\boldsymbol{r}-b\hat{\boldsymbol{z}}|}\right]. \tag{1}$$

The boundary condition (BC) is $V(\boldsymbol{r}) = 0$ for all $\boldsymbol{r}$ with $|\boldsymbol{r}| = R$. To find the two unknowns q' and b, we can consider the BC for two special cases, say $\boldsymbol{r} = \pm R\hat{\boldsymbol{z}}$. This gives

$$\text{for } \boldsymbol{r} = +R\hat{\boldsymbol{z}}: \qquad \frac{q}{|R-a|} + \frac{q'}{|R-b|} = 0 \quad \Rightarrow \quad q' = -\frac{R-b}{a-R}q, \tag{2}$$

$$\text{for } \boldsymbol{r} = -R\hat{\boldsymbol{z}}: \qquad \frac{q}{|-R-a|} + \frac{q'}{|-R-b|} = 0 \quad \Rightarrow \quad q' = -\frac{R+b}{R+a}q. \tag{3}$$

where we used that $a > R$ and $b < R$. Equating the two expressions for q' gives

$$(R+a)(R-b) = (R+b)(a-R) \quad \Rightarrow \quad 2R^2 = 2ab \quad \Rightarrow \quad b = \frac{R^2}{a}. \tag{4}$$

Inserting this result for b into one of the equations for q', say Eq. (3), gives

$$q' = -\frac{R+R^2/a}{R+a}q = -\frac{R}{a}\cdot\frac{1+R/a}{R/a+1}q = -\frac{R}{a}q. \tag{5}$$

We should now check whether this solution for q' and b also satisfies the BC's for the general case $|\boldsymbol{r}| = R$ (after all, while getting a solution to our set of two linear equations (2)-(3) in two unknowns was mathematically guaranteed, it is a priori not obvious that we would get the *same* solution regardless of which two special points on the spherical surface we selected). To this end, let us write

$$|\boldsymbol{r}-c\hat{\boldsymbol{z}}| = \sqrt{(\boldsymbol{r}-c\hat{\boldsymbol{z}})\cdot(\boldsymbol{r}-c\hat{\boldsymbol{z}})} = \sqrt{r^2 - 2c\boldsymbol{r}\cdot\hat{\boldsymbol{z}} + c^2} = \sqrt{r^2 - 2rc\cos\theta + c^2}. \tag{6}$$

Using this result, the second term inside the square brackets in (1) becomes, for $|\boldsymbol{r}| = R$,

$$\frac{-qR/a}{\sqrt{R^2 - 2R\cdot(R^2/a)\cos\theta + (R^2/a)^2}} = -\frac{q}{\sqrt{R^2 - 2Ra\cos\theta + a^2}}, \tag{7}$$

which is the negative of the first term, confirming the BC for an arbitrary point on the spherical surface.

(b) The surface charge density σ is given by

$$\sigma = -\epsilon_0\left[\left.\frac{\partial V}{\partial n}\right|_{\text{outside}} - \left.\frac{\partial V}{\partial n}\right|_{\text{inside}}\right] = -\epsilon_0\left.\frac{\partial V}{\partial n}\right|_{\text{outside}}. \tag{8}$$

Here "outside" ("inside") refer to evaluating the derivatives just outside (inside) the spherical surface. The "inside" term vanishes since the sphere is a conductor and thus an equipotential in electrostatics. Since the surface normal has the same direction as $\hat{\boldsymbol{r}}$, it follows that $\partial/\partial n = \partial/\partial r$. Thus

$$\begin{aligned}
\sigma &= -\frac{1}{4\pi}\frac{\partial}{\partial r}\left[\frac{q}{\sqrt{r^2-2ra\cos\theta+a^2}} + \frac{q'}{\sqrt{r^2-2rb\cos\theta+b^2}}\right]\Bigg|_{r=R} \\
&= \frac{1}{4\pi\epsilon_0}\left[\frac{q(R-a\cos\theta)}{(R^2-2Ra\cos\theta+a^2)^{3/2}} + \frac{q'(R-b\cos\theta)}{(R^2-2Rb\cos\theta+b^2)^{3/2}}\right] \\
&= \frac{q}{4\pi}\frac{R^2-a^2}{R(R^2+a^2-2Ra\cos\theta)^{3/2}}.
\end{aligned} \tag{9}$$

As is reasonable, this expression for σ has the opposite sign of q and its magnitude decreases with θ. Also, its dimension is [charge]/[length]^2, as it should be (it is good to make such checks).

The total charge of the entire system (point charge + sphere) is $q + Q \equiv Q_{\text{tot}}$. Here, Q_{tot} is also the charge appearing in the monopole term $Q_{\text{tot}}/4\pi\epsilon_0 r$ in the multipole expansion of the potential. From (1) one can see that the monopole term is $(q+q')/4\pi\epsilon_0 r$, so $Q_{\text{tot}} = q + q'$, giving

$$Q = Q_{\text{tot}} - q = (q + q') - q = q'. \tag{10}$$

Alternatively, Q can be found by integrating the surface charge density σ over the spherical surface:

$$Q = \int \sigma da = R^2 \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin\theta \, \sigma = \frac{qR(R^2 - a^2)}{2} \int_{-1}^{1} \frac{dx}{(R^2 + a^2 - 2Rax)^{3/2}} \tag{11}$$

(here the φ-integral just gave a factor 2π and we changed integration variables from θ to $x = \cos\theta$). The integral is $\int_{-1}^{1} dx \, (C + Dx)^{-3/2}$ with constants $C = R^2 - a^2$ and $D = -2Ra$. Changing integration variable to $u = C + Dx$, the integral becomes

$$\frac{1}{D} \int_{C-D}^{C+D} du \, u^{-3/2} = \frac{1}{D} \cdot \frac{1}{-3/2 + 1} u^{-3/2+1} \Bigg|_{C-D}^{C+D} = -\frac{2}{D} \left[\frac{1}{\sqrt{C+D}} - \frac{1}{\sqrt{C-D}} \right]. \tag{12}$$

Using $\sqrt{C \pm D} = \sqrt{R^2 + a^2 \mp 2Ra} = \sqrt{(R \mp a)^2} = a \mp R$, we get

$$Q = \frac{qR(R^2 - a^2)}{2} \cdot \frac{(-2)}{(-2Ra)} \underbrace{\left[\frac{1}{a - R} - \frac{1}{a + R} \right]}_{2R/(a^2 - R^2)} = -q\frac{R}{a} = q'. \tag{13}$$

(c) Call the second image charge q''. Since q and q' together make $V = 0$ at $r = R$, the job of q'' is to raise the potential from 0 to V_0 at $r = R$. Since all points with $|\boldsymbol{r}| = R$ should be raised by the same value V_0, q'' must be positioned equally far away from all these points, and therefore it must be placed at the origin $r = 0$. Its potential at $r = R$ is therefore $q''/4\pi\epsilon_0 R$. This should equal V_0, so $q'' = 4\pi\epsilon_0 R V_0$. The potential outside the sphere is $V(\boldsymbol{r}) = (4\pi\epsilon_0)^{-1}(q/|\boldsymbol{r} - a\hat{\boldsymbol{z}}| + q'/|\boldsymbol{r} - b\hat{\boldsymbol{z}}| + q''/|\boldsymbol{r}|)$.