TFY4240

Solution problem set 3 Autumn 2015

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Problem 1.

a) We have a function $f(x)$ on the interval $[-1, 1]$. Since the Legendre polynomials form a complete set on this interval, it is possible to write $f(x)$ as a linear combination of Legendre polynomials:

$$f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x). \tag{1}$$

Multiplying by P_m on both sides and integrating from -1 to 1 gives

$$\int_{-1}^{1} dx f(x) P_m(x) = \int_{-1}^{1} dx \sum_{\ell=0}^{\infty} A_\ell P_\ell(x) P_m(x). \tag{2}$$

Using the orthogonality of the Legendre polynomials, *i.e.*

$$\int_{-1}^{1} dx\, P_m(x) P_n(x) = \frac{2}{2n+1}\delta_{mn}, \tag{3}$$

and interchanging the order of summation and integration in Eq. (2) leads to the relation

$$\int_{-1}^{1} dx\, f(x) P_m(x) = A_m \frac{2}{2m+1}, \tag{4}$$

which is readily solved for the coefficients A_ℓ to give (after renaming m to ℓ)

$$A_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} dx\, f(x) P_\ell(x). \tag{5}$$

b)

$$f(x) = \begin{cases} -1 & x < 0 \\ +1 & x > 0 \end{cases} \tag{6}$$

Since $f(x)$ is an odd function, and the Legendre polynomial $P_\ell(x)$ is an even function for even ℓ, the integrand in (5) is odd for even ℓ, and thus A_ℓ vanishes for this case. Hence only Legendre polynomials of odd order are needed, *i.e.*

$$f(x) = \sum_{n=0}^{\infty} A_{2n+1} P_{2n+1}(x). \tag{7}$$

c) For odd ℓ, the integral in equation (5) becomes twice the integral from 0 to 1, which gives $A_\ell = (2\ell+1)\int_0^1 dx P_\ell(x)$. The first few coefficients are

$$A_1 = 3\int_0^1 dx\, x = \frac{3}{2}, \tag{8}$$

$$A_3 = 7\int_0^1 dx\, \frac{1}{2}(5x^3 - 3x) = \frac{7}{2}\left(\frac{5}{4} - \frac{3}{2}\right) = -\frac{7}{8}, \tag{9}$$

$$A_5 = 11\int_0^1 dx\, \frac{1}{8}(63x^5 - 70x^3 + 15x) = \frac{11}{8}\left(\frac{63}{6} - \frac{70}{4} + \frac{15}{2}\right) = \frac{11}{16}. \tag{10}$$

Problem 2

(a) Since the potential on the sphere surface is independent of the azimuthal angle φ, the problem has azimuthal symmetry. We can therefore expand the potential as

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Since $V(r,\theta)$ should $\to 0$ as $r \to \infty$, A_l must be 0 for all l (including $l=0$)

$$\Rightarrow V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \qquad (*)$$

To find the coefficients B_l, we need to consider the expansion $(*)$ for $r=R$ and use that it should equal $V(R,\theta) = V_0 \cos^2\theta$. The coefficients could then be determined by using the orthogonality relations for Legendre polynomials. However, because in this example $V(R,\theta) = V_0 \cos^2\theta$ has a simple expression in terms of Legendre polynomials, we can use that to simply read off the coefficients. To this end, we note that

$$P_0(\cos\theta) = 1 \quad , \quad P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$

$$\Rightarrow \cos^2\theta - \frac{2}{3} P_2(\cos\theta) = \frac{1}{3} = \frac{1}{3} P_0(\cos\theta)$$

$$\Rightarrow V(R,\theta) = \frac{V_0}{3}\left[P_0(\cos\theta) + 2\, P_2(\cos\theta)\right]$$

Comparing this with (*) evaluated at $r = R$, we can read off

$$l=0 \text{ term}: \quad \frac{B_0}{R} = \frac{V_0}{3} \quad \Rightarrow \quad B_0 = \frac{V_0 R}{3}$$

$$l=2 \text{ term}: \quad \frac{B_2}{R^3} = \frac{2V_0}{3} \quad \Rightarrow \quad B_2 = \frac{2V_0 R^3}{3}$$

$$l \neq 0,2: \quad \frac{B_l}{R^{l+1}} = 0 \quad \Rightarrow \quad B_l = 0$$

Inserting these results back into (*) gives

$$V(r,\theta) = \frac{V_0}{3}\left[\frac{R}{r}\underbrace{P_0(\cos\theta)}_{=1} + 2\left(\frac{R}{r}\right)^3 P_2(\cos\theta)\right]$$

Hence the potential outside the sphere ($r \geq R$) is

$$\underline{\underline{V_{\text{outside}}(r,\theta) = \frac{V_0}{3}\left[\frac{R}{r} + 2\left(\frac{R}{r}\right)^3 P_2(\cos\theta)\right]}}$$

Let us check that this expression reduces to the correct result at $r = R$:

$$V_{\text{outside}}(R,\theta) = \frac{V_0}{3}\left[1 + 2P_2(\cos\theta)\right]$$

$$= \frac{V_0}{3}\left[1 + 2\cdot\frac{1}{2}(3\cos^2\theta - 1)\right] = \underline{V_0\cos^2\theta} \quad \text{OK}$$

(b) The surface charge density σ on the spherical surface $r = R$ is given by

$$\sigma = -\epsilon_0 \left[\left.\frac{\partial V_{outside}}{\partial n}\right|_{r=R} - \left.\frac{\partial V_{inside}}{\partial n}\right|_{r=R} \right]$$

On the spherical surface, $\hat{n} = \hat{r}$, so the normal derivative is

expression for ∇ in spherical coordinates

$$\frac{\partial}{\partial n} \equiv \hat{n} \cdot \nabla = \hat{r} \cdot \nabla$$

$$= \hat{r} \cdot \left[\hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\varphi} \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \right]$$

$$= \frac{\partial}{\partial r} \qquad (\text{since } \hat{r} \cdot \hat{\theta} = \hat{r} \cdot \hat{\varphi} = 0)$$

$$\Rightarrow \quad \sigma = -\epsilon_0 \left[\left.\frac{\partial V_{outside}}{\partial r}\right|_{r=R} - \left.\frac{\partial V_{inside}}{\partial r}\right|_{r=R} \right]$$

Since we have found $V_{outside}(r, \theta)$, we can find the first term. What about the second term? If the sphere had been a conductor, it would have been 0, since the electric field $E = 0$ in a conductor. But the sphere is not a conductor, because $V(R, \theta)$ is not a constant (it depends on θ). So in order to calculate the second term we first have to find $V(r, \theta)$ inside the sphere.

The azimuthal symmetry again allows us to write

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

B_ℓ must be 0 for all ℓ; otherwise $V(r,\theta)$ would diverge as $r \to 0$

$$\Rightarrow \quad V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell \, r^\ell \, P_\ell(\cos\theta) \qquad (**)$$

Evaluating this at $r=R$ and comparing with $V(R,\theta) = \frac{V_0}{3}\left[P_0(\cos\theta) + 2P_2(\cos\theta)\right]$ gives

$\ell=0$ term: $\quad A_0 R^0 = \frac{V_0}{3} \quad \Rightarrow \quad A_0 = \frac{V_0}{3}$

$\ell=2$ term: $\quad A_2 R^2 = \frac{V_0}{3} \cdot 2 \Rightarrow A_2 = \frac{2V_0}{3R^2}$

$\ell \neq 0,2$: $\quad A_\ell R^\ell = 0 \quad \Rightarrow \quad A_\ell = 0$

Inserting these results back into $(**)$ gives

$$V_{inside}(r,\theta) = \frac{V_0}{3}\left[P_0(\cos\theta) + 2\left(\frac{r}{R}\right)^2 P_2(\cos\theta)\right]$$

(One can see that it reduces to $V_0 \cos^2\theta$ at $r=R$, as it should.)

To summarize, we have found

$$V_{outside}(r,\theta) = \frac{V_0}{3}\left[\frac{R}{r} + 2\left(\frac{R}{r}\right)^3 P_2(\cos\theta)\right]$$

$$V_{inside}(r,\theta) = \frac{V_0}{3}\left[1 + 2\left(\frac{r}{R}\right)^2 P_2(\cos\theta)\right]$$

This gives

$$\frac{\partial V_{outside}}{\partial r} = \frac{V_0}{3}\left[-\frac{R}{r^2} + 2\cdot 3\left(\frac{R}{r}\right)^2 \cdot \left(\frac{-R}{r^2}\right) P_2(\cos\theta)\right]$$

$$\Rightarrow \left.\frac{\partial V_{outside}}{\partial r}\right|_{r=R} = -\frac{V_0}{3R}\left[1 + 6\,P_2(\cos\theta)\right]$$

$$\frac{\partial V_{inside}}{\partial r} = \frac{V_0}{3} \cdot 2 \cdot 2 \frac{r}{R} \cdot \frac{1}{R} P_2(\cos\theta)$$

$$\Rightarrow \left.\frac{\partial V_{inside}}{\partial r}\right|_{r=R} = \frac{4V_0}{3R} P_2(\cos\theta)$$

The surface charge density is therefore

$$\sigma = -\epsilon_0 \frac{V_0}{3R} \left\{ -\left(1 + 6P_2(\cos\theta)\right) - 4P_2(\cos\theta) \right\}$$

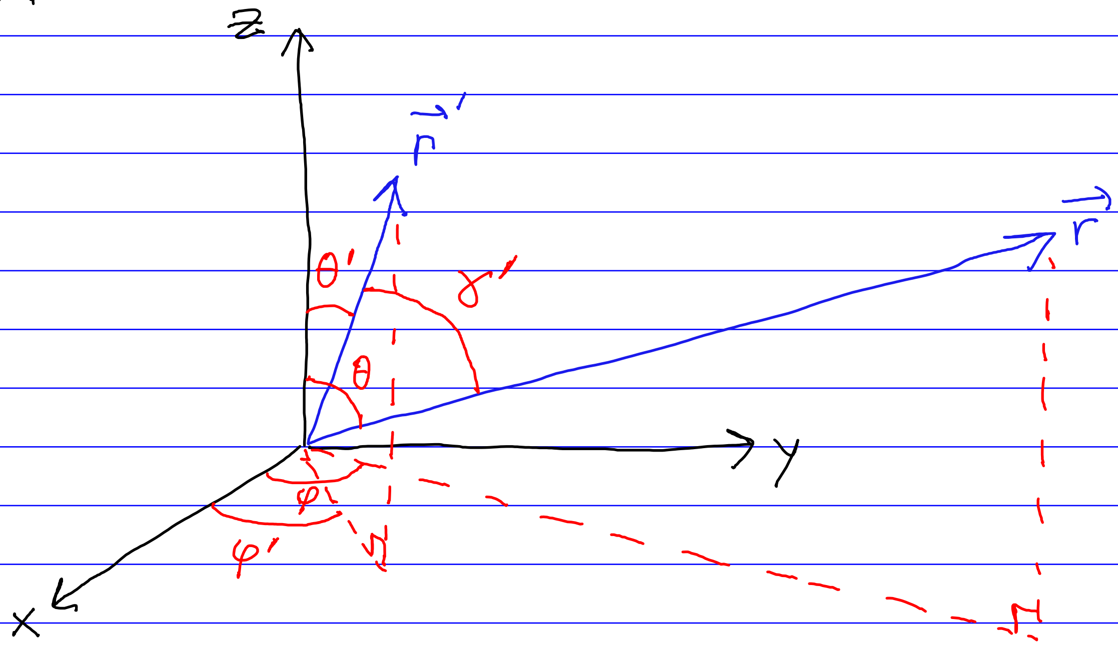
$$= \frac{\epsilon_0 V_0}{3R}\left[1 + 10 P_2(\cos\theta)\right] \equiv \sigma(\theta)$$

(c) In (a) we found $V_{outside}(r,\theta)$ from the potential $V(R,\theta)$ on the spherical surface. Now we are instead asked to find $V_{outside}(r,\theta)$ from the charge distribution (given by the surface charge density $\sigma(\theta)$ on the spherical surface). More precisely we are asked to find the quadrupole term in the multipole expansion for $V_{outside}$. We know this should be nonzero since there is a quadrupole contribution (i.e. $\propto 1/r^3$) in the expression we found for $V_{outside}$. (There is also a monopole contribution (i.e. $\propto 1/r$) which we could also alternatively find from the multipole expansion).

The quadrupole term in the multipole expansion is

$$V_{quad}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int d^3r' \, \rho(\vec{r}') (r')^2 P_2(\cos\gamma')$$

where γ' is the angle between $\vec{r}$ and $\vec{r}\,'$. In Griffiths and in lectures this angle was called θ', but I now wish to use θ' for the angle between $\hat{r}'$ and $\hat{z}$; see the figure.



Consider the integral in the quadrupole term. Using the expression for P_2, it reads

$$\frac{1}{2}\int d^3r'\,(r')^2\,\rho(\vec{r}\,')\left(3\cos^2\gamma' - 1\right) \qquad (\square)$$

Since

$$\cos\gamma' = \hat{r}\cdot\hat{r} = \hat{r}_i\,\hat{r}_i' \qquad \text{(summation convention)}$$

we get $\cos^2\gamma' = \hat{r}_i\,\hat{r}_j\,\hat{r}_i'\,\hat{r}_j'$,

so $(\square)$ can be written (cf. Problem 3.45 in Griffiths)

$$\frac{1}{2}\Big[\,3\,\hat{r}_i\,\hat{r}_j\int d^3r'\,(r')^2\,\rho(\vec{r}\,')\,\hat{r}_i'\,\hat{r}_j'$$
$$-\int d^3r'\,(r')^2\,\rho(\vec{r}\,')\Big]$$

Note that by rewriting in this way, all integrals are now independent of $\vec{r}$.

The charge density is given by

$$\rho(\vec{r}\,') = \sigma(\theta')\,\delta(r'-R).$$

Thus

$$\int d^3r'\,(r')^2\,\rho(\vec{r}\,') = \int \underbrace{dr'(r')^2\,d\varphi'\,\sin\theta'\,d\theta'}_{d^3r'}\,(r')^2\,\delta(r'-R)\,\sigma(\theta')$$

from $\int d\varphi'$

$$= \underbrace{2\pi}\,R^4 \int_0^{\pi} d\theta'\,\sin\theta'\,\underbrace{\sigma(\theta')} \qquad = \frac{\epsilon_0 V_0}{3R}\left[P_0(\cos\theta') + 10P_2(\cos\theta')\right]$$

$$= 2\pi R^4\,\frac{\epsilon_0 V_0}{3R}\int_0^{\pi} d\theta'\,\sin\theta'\left[P_0(\cos\theta') + 10\,P_2(\cos\theta')\right]$$

$$= \frac{2\pi\epsilon_0 V_0 R^3}{3}\cdot\int_{-1}^{1} dx\left[1 + 10\,\tfrac{1}{2}\left(3x^2-1\right)\right]$$

$$= \frac{2\pi\epsilon_0 V_0 R^3}{3}\left[2 + 15\,\tfrac{1}{3}\underbrace{x^3\Big|_{-1}^{1}}_{2} - 5\cdot 2\right] = \underline{\frac{4\pi\epsilon_0 V_0 R^3}{3}}$$

(As a check, note that this integral can be rewritten $R^2\int d^3r'\,\rho(\vec{r}\,')$ where $\int d^3r'\,\rho(\vec{r}\,')$ is the total charge in the monopole term ($\propto 1/r$) in $V_{outside}$. From this term we can read off the total charge as $4\pi\epsilon_0\frac{V_0 R}{3}$, agreeing with our result above)

Next consider the sum

$$\hat{r}_i \hat{r}_j \int d^3r' \, (r')^2 \, \rho(\vec{r}\,') \, \hat{r}_i' \, \hat{r}_j'$$

where
$$\hat{r}_x = \sin\theta \cos\varphi \, , \quad \hat{r}_x' = \sin\theta' \cos\varphi'$$
$$\hat{r}_y = \sin\theta \sin\varphi \, , \quad \hat{r}_y' = \sin\theta' \sin\varphi'$$
$$\hat{r}_z = \cos\theta \, , \quad \hat{r}_z' = \cos\theta'$$

There are $3 \cdot 3 = 9$ terms in the sum. Symbolically we can write the sum as (first letter i, second j)

$$xx + yy + zz + xy + yx$$
$$+ zx + zy + xz + yz$$

We note that $\rho(\vec{r}\,')$ is independent of φ'

$$\Rightarrow \quad zx, \, xz \propto \int_0^{2\pi} d\varphi' \cos\varphi' = 0$$

$$zy, \, yz \propto \int_0^{2\pi} d\varphi' \sin\varphi' = 0$$

$$xy, \, yx \propto \int_0^{2\pi} d\varphi' \cos\varphi' \sin\varphi' = 0$$

Furthermore (define the shorthand $K \equiv d^3r' (r')^2 \rho(\vec{r}\,')$)

$$xx + yy = \sin^2\theta \cos^2\varphi \int K \sin^2\theta' \cos^2\varphi'$$
$$+ \sin^2\theta \sin^2\varphi \int K \sin^2\theta' \sin^2\varphi'$$

Since the average of $\cos^2\varphi'$ and $\sin^2\varphi'$ in $[0, 2\pi)$ is $\frac{1}{2}$, we can replace both factors by $\frac{1}{2}$

$$\Rightarrow \quad xx + yy = \frac{1}{2} \sin^2\theta \overbrace{(\cos^2\varphi + \sin^2\varphi)}^{=1} \int K \sin^2\theta'$$
$$= \frac{1}{2} \sin^2\theta \int K \sin^2\theta'$$

$$= \frac{1}{2}(1-\cos^2\theta)\int K\,(1-\cos^2\theta')$$

Finally, $zz = \cos^2\theta \int K \cos^2\theta'$

So the sum reduces to

$$\frac{1}{2}(1-\cos^2\theta)\int K(1-\cos^2\theta') + \cos^2\theta\int K\cos^2\theta'$$

$$= \frac{1}{2}\left(\int K - \int K\cos^2\theta'\right)$$

$$+ \cos^2\theta\left(-\frac{1}{2}\int K + \frac{3}{2}\int K\cos^2\theta'\right)$$

We already calculated $\int K = \frac{4\pi\epsilon_0 V_0 R^3}{3} \equiv C$
so we just need to calculate

$$\int K\cos^2\theta' = \int d^3r'\,(r')^2\,\rho(\vec{r}\,')\cos^2\theta'$$

$$= \int dr'\,(r')^2 \sin\theta'\,d\theta'\,d\varphi'\,(r')^2\,\frac{\epsilon_0 V_0}{3R}\,\delta(r'-R)$$

$$\cdot\left(P_0(\cos\theta') + 10\,P_2(\cos\theta')\right)\underbrace{\left(\frac{1}{3}P_0(\cos\theta') + \frac{2}{3}P_2(\cos\theta')\right)}_{\cos^2\theta'}$$

$$= 2\pi\,\frac{\epsilon_0 V_0}{3R}\cdot R^4\cdot\frac{1}{3}\int_{-1}^{1} dx\,(P_0 + 10P_2)(P_0 + 2P_2)$$

$$= \frac{2\pi}{9}\epsilon_0 V_0 R^3\left[\frac{2}{2\cdot 0+1} + 10\cdot 2\cdot\frac{2}{2\cdot 2+1}\right] \leftarrow \text{using orthogonality relations for Legendre polynomials}$$

$$= \frac{4\pi}{9}\epsilon_0 V_0 R^3\,[1+4] = \frac{4\pi\epsilon_0 V_0 R^3}{3}\cdot\frac{5}{3} = \frac{5}{3}C$$

Thus the sum becomes

$$\frac{1}{2}C\left(1-\frac{5}{3}\right) + \frac{1}{2}\cos^2\theta\left(-C + 3\cdot\frac{5}{3}C\right) = \frac{C}{3}\left(6\cos^2\theta - 1\right)$$

Putting it all together, the quadrupole term becomes

$$\frac{1}{4\pi\epsilon_0}\frac{1}{2r^3}\left[3\cdot\frac{C}{3}\left(6\cos^2\theta-1\right)-C\right]$$

$$=\frac{1}{4\pi\epsilon_0}\frac{C}{r^3}\underbrace{\left(3\cos^2\theta-1\right)}_{2P_2(\cos\theta)}=\frac{1}{\cancel{4\pi\epsilon_0}}\frac{\cancel{4\pi\epsilon_0}V_0}{3}\left(\frac{R}{r}\right)^3 2P_2(\cos\theta)$$

$$=\frac{V_0}{3}\cdot 2\left(\frac{R}{r}\right)^2 P_2(\cos\theta)$$

which indeed agrees with the quadrupole contribution to $V_{outside}$ found in (a).

For the interested reader, I also present an alternative calculation that is more elegant and powerful (involving the full multipole expansion), but invokes a result involving spherical harmonics that we have not proved. It starts from the multipole expansion

$$V(\vec{r})=\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int d^3r'(r')^l\rho(\vec{r}')P_l(\cos\gamma')$$

Now use the result ("addition theorem for spherical harmonics")

$$P_l(\cos\gamma')=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

to get

$$V(\vec{r})=\frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y_{lm}(\theta,\varphi)}{(2l+1)r^{l+1}}\int d^3r'(r')^l\rho(\vec{r}')Y^*_{lm}(\theta',\varphi')$$

Next use that $Y_{lm}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\varphi}$

Since our $\rho(\vec{r}\,')$ is indep. of φ', the integral $\int d^3r'$ will be proportional to $\int_0^{2\pi} d\varphi' \exp(-im\varphi') = 2\pi\,\delta_{m,0}$. Thus only the $m=0$ term survives the integration, giving

$$V(\vec{r}) = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{2l+1}\frac{Y_{l0}(\theta,\varphi)}{r^{l+1}}\int d^3r'(r')^l \rho(\vec{r}\,')\, Y^*_{l0}(\theta',\varphi')$$

Now use $Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\, P_l^0(\cos\theta)$

and $P_l^0(\cos\theta) = P_l(\cos\theta)$, which gives

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{P_l(\cos\theta)}{r^{l+1}}\int d^3r'(r')^l \rho(\vec{r}\,')\, P_l(\cos\theta')}$$

This expression is valid when $\rho(\vec{r}\,')$ is independent of φ'. Note that although the integral looks superficially like the one discussed in Griffiths and the lectures, the meaning of θ' is different.

Now we will evaluate $V(\vec{r})$ using the above formula for the charge distribution in our case. The integral becomes

$$\int d^3r'(r')^l \rho(\vec{r}\,')\, P_l(\cos\theta')$$

$$= \int dr'(r')^2 d\theta' \sin\theta' d\varphi' (r')^l \frac{\epsilon_0 V_0}{3R}\left(P_0(\cos\theta') + 10 P_2(\cos\theta')\right) \cdot \delta(r'-R)\cdot P_l(\cos\theta')$$

$$= 2\pi R^{l+2}\frac{\epsilon_0 V_0}{3R}\int_{-1}^{1} dx\,\left(P_0(x) + 10 P_2(x)\right) P_l(x)$$

$$= \frac{2\pi\epsilon_0 V_0 R^{l+1}}{3} \left[\frac{2}{2\cdot 0+1} \delta_{l,0} + 10 \cdot \frac{2}{2\cdot 2+1} \delta_{l,2} \right]$$

$$= \frac{4\pi\epsilon_0 V_0 R^{l+1}}{3} \left(\delta_{l,0} + 2\,\delta_{l,2} \right)$$

This gives

$$V(\vec{r}) = \frac{1}{\cancel{4\pi\epsilon_0}} \sum_{l=0}^{\infty} \frac{P_l(\cos\theta)}{r^{l+1}} \cdot \frac{\cancel{4\pi\epsilon_0} V_0 R^{l+1}}{3} \left(\delta_{l,0} + 2\delta_{l,2} \right)$$

$$= \frac{V_0}{3} \left[\frac{R}{r} + 2\left(\frac{R}{r}\right)^3 P_2(\cos\theta) \right]$$

which agrees with our result for $V_{outside}$ in (a).

Problem 3

(a) See the solution to Problem 2(a) in Tutorial 2. (The equation that we effectively solve is the Poisson equation $\nabla^2 V(\vec{r}) = -\rho(\vec{r})/\epsilon_0$ in a volume with the charge density $\rho(\vec{r})$ known inside the volume and the potential $V(\vec{r})$ specified on the boundary of the volume.)

(b) $$V(\vec{r}) = \frac{q}{4\pi\epsilon_0 |\vec{r}-(R+h)\hat{z}|} - \frac{q}{4\pi\epsilon_0 |\vec{r}-(R-h)\hat{z}|}$$

The first term is the potential from the charge q located at $\vec{r}_q = (R+h)\hat{z}$. The second term is the potential from the image charge of q, which here has charge $\bar{q} = -q$ and is located at $\vec{r}_{\bar{q}} = (R-h)\hat{z}$.

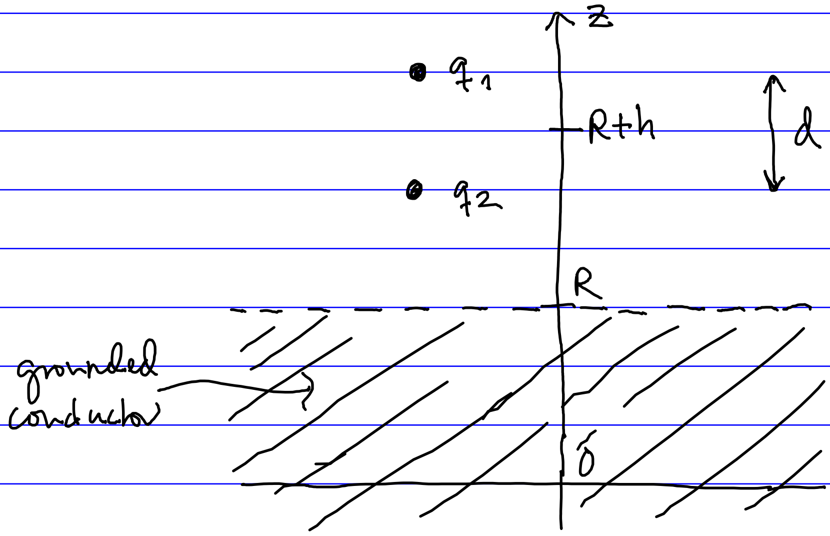
The vectors $\vec{\rho}_\pm = \vec{r} - (R \pm h)\hat{z}$ are the difference vectors between $\vec{r}$ and the location of the charge (upper sign) / image charge (lower sign).

Along the plane $z = R$ we can write $\vec{r} = \vec{r}_\parallel + R\hat{z}$ where $\vec{r}_\parallel$ is in the xy plane. Thus

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{r}_\parallel - h\hat{z}|} - \frac{1}{|\vec{r}_\parallel + h\hat{z}|}\right] = 0$$

when $\vec{r}$ is in the plane $z = R$, as required by the boundary condition for the grounded conductor.

(c) Sketch of configuration:



(In reality we should have $d \ll h$; I have drawn d bigger to make the drawing clearer)

The charges and their locations are:

$$q_1 = q \,, \quad \vec{r}_{q_1} = \left(R + h + \frac{d}{2}\right)\hat{z}$$

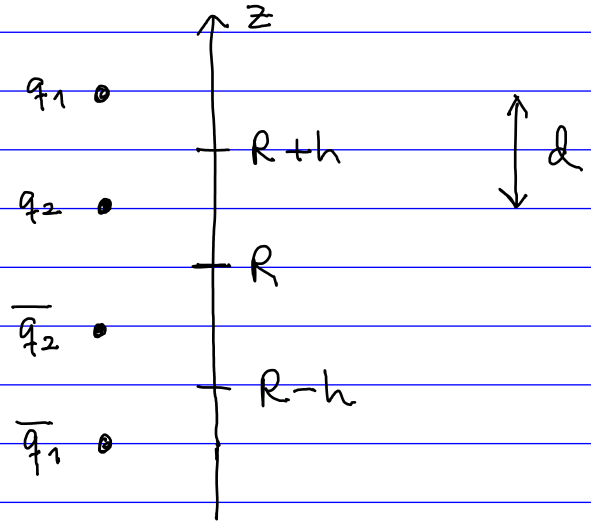
$$q_2 = -q \,, \quad \vec{r}_{q_2} = \left(R + h - \frac{d}{2}\right)\hat{z}$$

For each charge q_i we introduce an image charge $\overline{q}_i$. In this problem, $\overline{q}_i = -q_i$ and the position of the image charge $\overline{q}_i$ is mirror-symmetric about the plane $z = R$ with respect to the position of the original charge q_i

$$\Rightarrow \quad \overline{q}_1 = -q \,, \quad \vec{r}_{\overline{q}_1} = \left(R - h - \frac{d}{2}\right)\hat{z}$$

$$\overline{q}_2 = q \,, \quad \vec{r}_{\overline{q}_2} = \left(R - h + \frac{d}{2}\right)\hat{z}$$

This gives the alternative system <u>with</u> image charges but <u>without</u> the conductor, valid for calculating V in the region with $z \geq R$:



The potential $V(\vec{r})$ in the region $z \geq R$ is obtained by adding the potentials from the charges q_i and their image charges $\bar{q}_i$:

$$V(\vec{r}) = \sum_{i=1,2} \left(\frac{q_i}{4\pi\epsilon_0 |\vec{r} - \vec{r}_{q_i}|} + \frac{\bar{q}_i}{4\pi\epsilon_0 |\vec{r} - \vec{r}_{\bar{q}_i}|} \right)$$

$$= \frac{q}{4\pi\epsilon_0} \left[\frac{1}{|\vec{r} - (R+h+\frac{d}{2})\hat{z}|} - \frac{1}{|\vec{r} - (R-h-\frac{d}{2})\hat{z}|} - \frac{1}{|\vec{r} - (R+h-\frac{d}{2})\hat{z}|} + \frac{1}{|\vec{r} - (R-h+\frac{d}{2})\hat{z}|} \right]$$

We now make an expansion in $d/\rho_+ \ll 1$.
The 1st term:

$$\frac{1}{|\vec{r} - (R+h+\frac{d}{2})\hat{z}|} = \frac{1}{|\vec{\rho}_+ - \frac{\vec{d}}{2}|} = \frac{1}{\sqrt{(\vec{\rho}_+ - \vec{d}/2)^2}}$$

$$= \frac{1}{\sqrt{\rho_+^2 - \vec{\rho}_+\cdot\vec{d} + d^2/4}} = \frac{1}{\rho_+ \sqrt{1 - \frac{\vec{d}\cdot\vec{\rho}_+}{\rho_+^2} + \frac{d^2}{4\rho_+^2}}}$$

$$= \frac{1}{\rho_+}\left(1 - \left(-\frac{1}{2}\right)\frac{\vec{d}\cdot\vec{\rho}_+}{\rho_+^2} + O\left(\left(\frac{d}{\rho_+}\right)^2\right)\right) \approx \frac{1}{\rho_+} + \frac{1}{2}\frac{\vec{d}\cdot\vec{\rho}_+}{\rho_+^2}$$

The 3rd term: obtained from the 1st by letting $\vec{d} \to -\vec{d}$

$$\Rightarrow \frac{1}{|\vec{r} - (R + h - \frac{d}{2})\hat{z}|} \approx \frac{1}{\rho_+} - \frac{1}{2} \frac{\vec{d} \cdot \hat{\rho}_+}{\rho_+^2}$$

The 2nd term: obtained from the 3rd by letting $h \to -h$, i.e. $\vec{\rho}_+ \to \vec{\rho}_-$

$$\Rightarrow \frac{1}{|\vec{r} - (R - h - \frac{d}{2})\hat{z}|} \approx \frac{1}{\rho_-} - \frac{1}{2} \frac{\vec{d} \cdot \hat{\rho}_-}{\rho_-^2}$$

(this is an expansion in d/ρ_-; note that since $d/\rho_+ \ll 1$ by assumption, then we also have $d/\rho_- \ll 1$ since $\rho_- \geq \rho_+$ because $\vec{r}$ lies above (or in) the plane $z = R$).

The 4th term: obtained from the 2nd by letting $\vec{d} \to -\vec{d}$

$$\Rightarrow \frac{1}{|\vec{r} - (R - h + \frac{d}{2})\hat{z}|} \approx \frac{1}{\rho_-} + \frac{1}{2} \frac{\vec{d} \cdot \hat{\rho}_-}{\rho_-^2}$$

This gives

$$V(\vec{r}) \approx \frac{q}{4\pi\epsilon_0}\left[\left(\cancel{\frac{1}{\rho_+}} + \frac{1}{2}\frac{\vec{d} \cdot \hat{\rho}_+}{\rho_+^2}\right) - \left(\cancel{\frac{1}{\rho_-}} - \frac{1}{2}\frac{\vec{d} \cdot \hat{\rho}_-}{\rho_-^2}\right) - \left(\cancel{\frac{1}{\rho_+}} - \frac{1}{2}\frac{\vec{d} \cdot \hat{\rho}_+}{\rho_+^2}\right) + \left(\cancel{\frac{1}{\rho_-}} + \frac{1}{2}\frac{\vec{d} \cdot \hat{\rho}_-}{\rho_-^2}\right)\right]$$

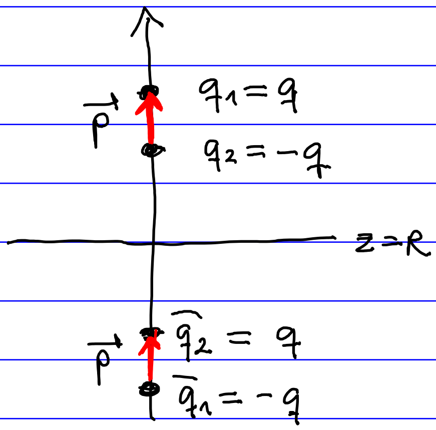
$$= \frac{q\vec{d} \cdot \hat{\rho}_+}{4\pi\epsilon_0 \rho_+^2} + \frac{q\vec{d} \cdot \hat{\rho}_-}{4\pi\epsilon_0 \rho_-^2} = \underline{\underline{\frac{\vec{p} \cdot \hat{\rho}_+}{4\pi\epsilon_0 \rho_+^2} + \frac{\vec{p} \cdot \hat{\rho}_-}{4\pi\epsilon_0 \rho_-^2}}}$$

which is valid for $z \geq R$ and to leading order in the small parameters d/ρ_+, d/ρ_-.

The first term is the potential of an electric dipole with dipole moment $\vec{p}$ at position $(R+h)\hat{z}$ (this is the potential from $q_1 = q$ and $q_2 = -q$).

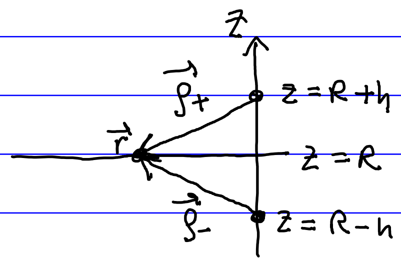
The second term is the potential of an electric dipole with dipole moment $\vec{p}$ at position $(R-h)\hat{z}$ (this is the potential from $\bar{q}_1 = -q$ and $\bar{q}_2 = -q$).

Note that the electric dipole moment $\vec{p}$ of the two image charges is the same (including the direction) as that of the original charges. This is also easy to see just from the charge configuration:



(the direction of the electric dipole moment is from the negative charge to the positive charge; here we have assumed $q > 0$ for concreteness, which gives $\vec{p}$ pointing in the $+\hat{z}$

Let us check that $V = 0$ when $z = R$:



At $z = R$, $\rho_+ = \rho_- \equiv \rho$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0 \rho^2} \vec{p} \cdot (\hat{\rho}_+ + \hat{\rho}_-) = 0$$

since $\vec{p} = |\vec{p}|\,\hat{z}$ and $\hat{\rho}_+ + \hat{\rho}_-$ has z-component 0, because $(\hat{\rho}_+)_z = -(\hat{\rho}_-)_z$ when $z = R$.

(d) By symmetry, the image charge $\bar{q}$ must be positioned on the z axis. We can therefore write its position as $\vec{r}_{\bar{q}} = z_{\bar{q}}\hat{z}$, where $|z_{\bar{q}}| < R$ since the image charge must be outside the region of interest (the outside of the sphere). Also let us write $\vec{r}_q \equiv z_q \hat{z}$ where $z_q = R + h$. The total potential outside the sphere can then be written as

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0 |\vec{r} - \vec{r}_q|} + \frac{\bar{q}}{4\pi\epsilon_0 |\vec{r} - \vec{r}_{\bar{q}}|}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|r\hat{r} - z_q\hat{z}|} + \frac{\bar{q}}{|r\hat{r} - z_{\bar{q}}\hat{z}|}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r|\hat{r} - \frac{z_q}{r}\hat{z}|} + \frac{\bar{q}}{|z_{\bar{q}}||\hat{z} - \frac{r}{z_{\bar{q}}}\hat{r}|}\right]$$

The boundary condition is $V(\vec{r}) = 0$ at $r = R$, i.e.

$$\frac{q}{R|\hat{r} - \frac{z_q}{R}\hat{z}|} + \frac{\bar{q}}{|z_{\bar{q}}||\hat{z} - \frac{R}{z_{\bar{q}}}\hat{r}|} = 0$$

This is satisfied if

$$\frac{q}{R} = -\frac{\bar{q}}{|z_{\bar{q}}|} \quad \text{and} \quad \frac{z_q}{R} = \frac{R}{z_{\bar{q}}}$$

(the latter condition ensures $|\hat{r} - \frac{z_q}{R}\hat{z}| = |\hat{z} - \frac{R}{z_{\bar{q}}}\hat{r}|$:

$$|\hat{r} - \frac{z_q}{R}\hat{z}| = \sqrt{1 - 2\frac{z_q}{R}\hat{r}\cdot\hat{z} + \left(\frac{z_q}{R}\right)^2} = \sqrt{1 - 2\frac{R}{z_{\bar{q}}}\hat{r}\cdot\hat{z} + \left(\frac{R}{z_{\bar{q}}}\right)^2}$$

$$= |\hat{z} - \frac{R}{q}\hat{r}|\ )$$

$$\Rightarrow \quad \underline{z_{\bar{q}}} = \frac{R^2}{z_q} = \underline{\frac{R^2}{R+h} = R\frac{1}{1+h/R}}$$

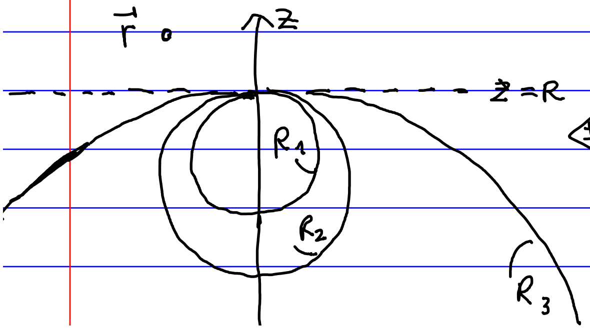
$$\underline{\bar{q}} = -\frac{|z_{\bar{q}}|}{R}q = -\frac{R}{z_q}q = -\frac{R}{R+h}q = \underline{-\frac{1}{1+h/R}q}$$

This solution method was based on a "clever" rewriting of $V(\vec{r})$. Alternatively, one can solve the two equations $V(\vec{r})\big|_{\vec{r}=r\hat{z}} = 0$ and $V(\vec{r})\big|_{\vec{r}=-R\hat{z}} = 0$ (as done in the lectures) for the two unknowns $\bar{q}$ and $z_{\bar{q}}$, and then check that this also gives $V(\vec{r})=0$ for all other points on the sphere. See also the method outlined in Problem 3.7 in Griffiths, 3rd ed. (Problem 3.8 in the 4th ed.)

Inserting for $\bar{q}$ and $\vec{r}_{\bar{q}}$, the potential outside the sphere becomes

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{r}-(R+h)\hat{z}|} - \frac{\frac{1}{1+h/R}}{|\vec{r}-R\frac{1}{1+h/R}\hat{z}|}\right]$$

(e) If we let the sphere get bigger ($R\to\infty$), while keeping other parameters fixed, in the way indicated in the figure below, the configuration with a charge outside a grounded sphere will approach the configuration with a charge outside (above) a grounded half-space:



⇐ As the radius R increases, the spherical surface approaches the plane $z=R$

This corresponds to also restricting $\vec{r}$ to positions

$$\vec{r} = \vec{r}_{\parallel} + (R + \Delta z)\,\hat{z} \qquad \text{with } \Delta z \geq 0$$

and then letting $R \to \infty$ while $\vec{r}_{\parallel}$, Δz and h are kept fixed. Let us check that in this limit the potential outside the sphere reduces to the potential above the infinite flat conducting plane:

We get

$$\vec{r} - (R+h)\hat{z} = \vec{r}_{\parallel} + (R+\Delta z)\hat{z} - (R+h)\hat{z} = \underline{\vec{r}_{\parallel} + (\Delta z - h)\hat{z}}$$

(independent of R, and thus constant as $R \to \infty$),

$$\vec{r} - R\,\frac{1}{1+h/R}\,\hat{z} = \vec{r}_{\parallel} + (R+\Delta z)\hat{z} - R\,\frac{1}{1+h/R}\,\hat{z}$$

$$\approx \vec{r}_{\parallel} + (R+\Delta z)\hat{z} - R\left(1 - \frac{h}{R}\right)\hat{z} \qquad \text{as } R \to \infty$$

$$= \underline{\vec{r}_{\parallel} + (\Delta z + h)\hat{z}} \qquad \text{as } R \to \infty$$

Also, the factor $1/(1+h/R)$ in the image charge $\to 1$ as $R \to \infty$

$$\Rightarrow V(\vec{r}) \xrightarrow{R \to \infty} \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec{r}_{\parallel} + (\Delta z - h)\hat{z}|} - \frac{1}{|\vec{r}_{\parallel} + (\Delta z + h)\hat{z}|}\right]$$

which indeed agrees with the expression for $V(\vec{r})$ found for the system studied in (b).

(f) (It is implicit that both charges are outside the sphere, i.e. $d/2 < h$.) We can use the results found in (d) to obtain $V(\vec{r})$ for this case too. For each charge q_i, at position $\vec{r}_{q_i} = z_{q_i}\hat{z}$, we have an image charge $\bar{q}_i$ at position $\vec{r}_{\bar{q}_i} = z_{\bar{q}_i}\hat{z}$, where

$$\bar{q}_i = -\frac{R}{z_{q_i}} q_i \quad , \quad z_{\bar{q}_i} = \frac{R^2}{z_{q_i}}$$

The potential $V(\vec{r})$ outside the sphere is then

$$\underline{\underline{V(\vec{r}) = \sum_{i=1,2}\left(\frac{q_i}{4\pi\epsilon_0|\vec{r}-\vec{r}_{q_i}|} + \frac{\bar{q}_i}{4\pi\epsilon_0|\vec{r}-\vec{r}_{\bar{q}_i}|}\right)}} \quad (*)$$

The monopole term in the multipole expansion for $V(\vec{r})$ is $Q/4\pi\epsilon_0 r$ where Q is the total charge due to all charges in $(*)$, i.e.

$$Q = q_1 + q_2 + \bar{q}_1 + \bar{q}_2$$

Using $q_1 = q$, $q_2 = -q$,

$$\bar{q}_1 = -\frac{R}{z_{q_1}} q_1 = -\frac{R}{R+h+d/2} q,$$

$$\bar{q}_2 = -\frac{R}{z_{q_2}} q_2 = \frac{R}{R+h-d/2} q,$$

we get

$$\underline{\underline{Q = \bar{q}_1 + \bar{q}_2 = qR\left(\frac{1}{R+h-d/2} - \frac{1}{R+h+d/2}\right)}}$$

Since the expression inside the parenthesis is > 0, Q has the same sign as q, i.e. positive.

Q is the total charge of the system. Since the system consists of the charges q_1 and q_2 and the conducting sphere, it follows that

$$Q = q_1 + q_2 + q_s$$

where q_s is the charge induced on the surface of the conducting sphere. Since $q_2 = -q_1$,

$$\underline{\underline{Q = q_s}}$$

This is the physical interpretation of Q.

We found $Q > 0$. This can be qualitatively understood as follows: The two charges q_1 and q_2 will each induce surface charges on the sphere with opposite sign of q_i. And the closer to the sphere the charge is, the bigger the magnitude of the induced surface charge. Since the negative charge q_2 is closer to the sphere than the positive charge q_1, the net induced surface charge q_s is positive.