TFY4240
Solution problem set 3

NTNU
Institutt for fysikk

Problem 1.
See handwritten solution further back.

Problem 2.
See Griffiths.

Problem 3.
See `http://www.physicspages.com/2012/01/10/laplaces-equation-fourier-series-examples-3-three-dimensions/`

Problem 4.

a) We have a function $f(x)$ on the interval $[-1, 1]$. Since the Legendre polynomials form a complete set on this interval, it is possible to write $f(x)$ as a linear combination of Legendre polynomials:

$$f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x). \tag{1}$$

Multiplying by P_m on both sides and integrating from -1 to 1 gives

$$\int_{-1}^{1} dx f(x)P_m(x) = \int_{-1}^{1} dx \sum_{\ell=0}^{\infty} A_\ell P_\ell(x)P_m(x). \tag{2}$$

Using the orthogonality of the Legendre polynomials, *i.e.*

$$\int_{-1}^{1} dx\, P_m(x)P_n(x) = \frac{2}{2n+1}\delta_{mn}, \tag{3}$$

and interchanging the order of summation and integration in Eq. (2) leads to the relation

$$\int_{-1}^{1} dx\, f(x)P_m(x) = A_m \frac{2}{2m+1}, \tag{4}$$

which is readily solved for the coefficients A_ℓ to give (after renaming m to ℓ)

$$A_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} dx\, f(x)P_\ell(x). \tag{5}$$

b)

$$f(x) = \begin{cases} -1 & x < 0 \\ +1 & x > 0 \end{cases} \tag{6}$$

Since $f(x)$ is an odd function, and the Legendre polynomial $P_\ell(x)$ is an even function for even ℓ, the integrand in (5) is odd for even ℓ, and thus A_ℓ vanishes for this case. Hence only Legendre polynomials of odd order are needed, *i.e.*

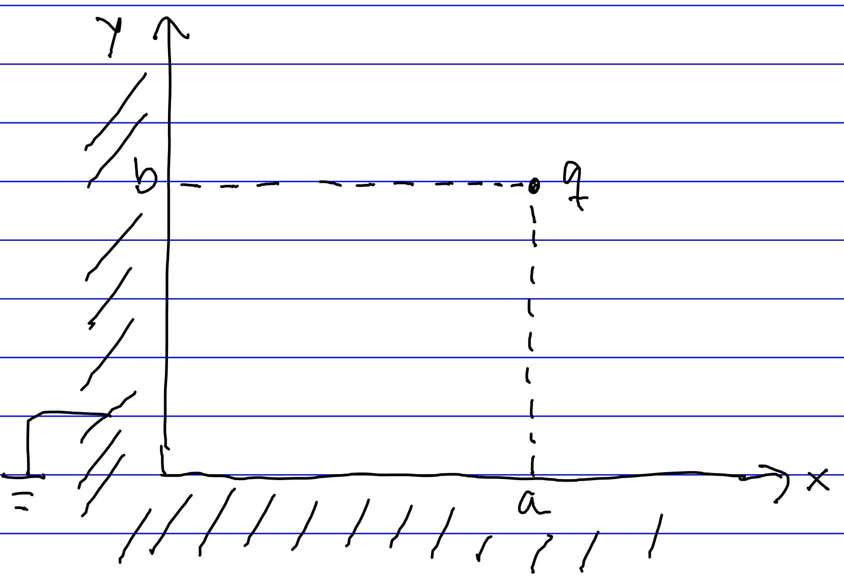
$$f(x) = \sum_{n=0}^{\infty} A_{2n+1} P_{2n+1}(x). \tag{7}$$

c) For odd ℓ, the integral in equation (5) becomes twice the integral from 0 to 1, which gives $A_\ell = (2\ell + 1) \int_0^1 dx P_\ell(x)$. The first few coefficients are

$$\begin{aligned} A_1 &= 3 \int_0^1 dx\, x = \frac{3}{2}, && (8) \\ A_3 &= 7 \int_0^1 dx\, \frac{1}{2}(5x^3 - 3x) = \frac{7}{2}\left(\frac{5}{4} - \frac{3}{2}\right) = -\frac{7}{8}, && (9) \\ A_5 &= 11 \int_0^1 dx\, \frac{1}{8}(63x^5 - 70x^3 + 15x) = \frac{11}{8}\left(\frac{63}{6} - \frac{70}{4} + \frac{15}{2}\right) = \frac{11}{16}. && (10) \end{aligned}$$

Problem 1

(a)



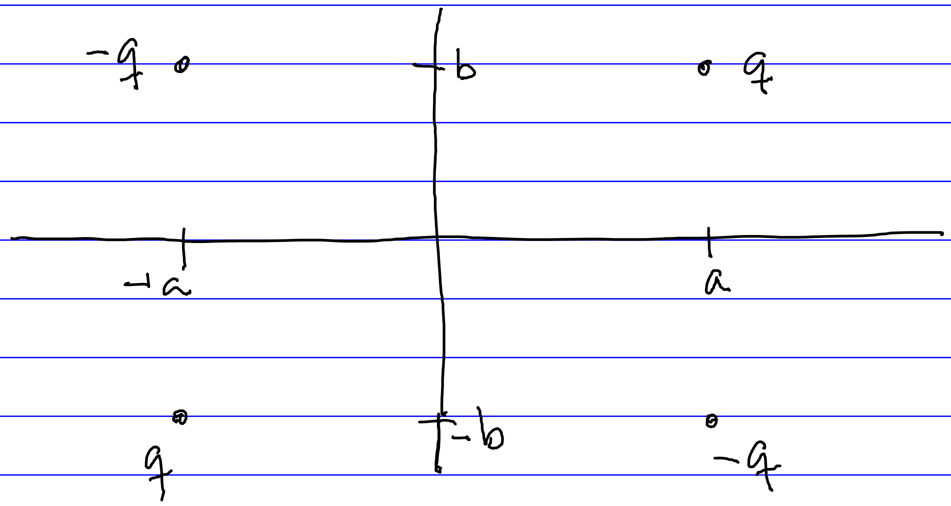
We try to use a similar strategy as done for a charge above a single conducting plane

First, put an image charge $-q$ at $(x,y) = (a,-b)$. This cancels V_q from q in the horizontal plane (xz plane, i.e. $y=0$)

Second, put an image charge $-q$ at $(x,y) = (-a,b)$. This cancels V_q from q in the vertical plane (yz plane, i.e. $x=0$).

However, these two image charges also mess up the potential on the "other" plane: the first image charge makes $V \neq 0$ in the yz plane, and the second image charge makes $V \neq 0$ in the xz plane.

However, these messes can be cleaned up by putting a third image charge $+q$ at $(x, y) = (-a, -b)$. Thus we have the following charge configuration:



(Note the essential fact that all image charges are placed outside the region $x>0$, $y>0$ where we want to find V.)

The potential $V(x, y, z)$ at an arbitrary point (x, y, z) outside the conductor (i.e. $x \geq 0$ and $y \geq 0$) is thus the sum of the potentials from each charge:

$$V(x, y, z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+(y-b)^2+z^2}} - \frac{1}{\sqrt{(x-a)^2+(y+b)^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+(y-b)^2+z^2}} + \frac{1}{\sqrt{(x+a)^2+(y+b)^2+z^2}}\right]$$

You can see that $V=0$ for $x=0$ and for $y=0$. Thus V vanishes everywhere on the conducting planes, as required by the boundary conditions.

(b) We wish to find the force $\vec{F}_q$ on the charge q. Note that this is not simply given by

$$(*) \quad \vec{F}_q = q\vec{E} \quad \text{with } \vec{E} = -\nabla V \text{ evaluated at the position of } q$$

The reason is that V also contains a contribution from q itself, and this would give a force on q from itself if we applied $(*)$. Thus we need to exclude this "self-force". This is done simply by omitting the contribution from q in V. Thus let V' be the remaining terms in V (ie the terms from all image charges). This gives

$$\vec{F}_q = q\vec{E}' \quad \text{with } \vec{E}' = -\nabla V'$$

i.e.

$$\vec{F}_q = -q\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\frac{q}{4\pi\epsilon_0}\left[-\frac{1}{\sqrt{(x-a)^2+(y+b)^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+(y-b)^2+z^2}} + \frac{1}{\sqrt{(x+a)^2+(y+b)^2+z^2}}\right]\Bigg|_{(x=a,\,y=b,\,z=0)}$$

The z-component can be seen to be prop. to z and thus vanishes since z should be set to zero at the end. So we omit the z component in the following

$$\vec{F}_q = -\frac{q^2}{4\pi\epsilon_0}\left(-\frac{1}{2}\right)\Bigg\{ \hat{x}\Bigg[-\frac{2(x-a)}{[(x-a)^2+(y+b)^2+z^2]^{3/2}}$$

$$-\frac{2(x+a)}{[(x+a)^2+(y-b)^2+z^2]^{3/2}} + \frac{2(x+a)}{[(x+a)^2+(y+b)^2+z^2]^{3/2}}\Bigg]$$

$$+\hat{y}\Bigg[-\frac{2(y+b)}{[(x-a)^2+(y+b)^2+z^2]^{3/2}} - \frac{2(y-b)}{[(x+a)^2+(y-b)^2+z^2]^{3/2}}$$

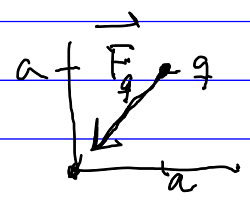
$$+\frac{2(y+b)}{[(x+a)^2+(y+b)^2+z^2]^{3/2}}\Bigg]\Bigg\}\Bigg|_{(x=a,\,y=b,\,z=0)}$$

$$= \frac{q^2}{4\pi\epsilon_0}\Bigg\{ -\hat{x}\,2a\left[\frac{1}{(2a)^3} - \frac{1}{[(2a)^2+(2b)^2]^{3/2}}\right]$$

$$-\hat{y}\,2b\left[\frac{1}{(2b)^3} - \frac{1}{[(2a)^2+(2b)^2]^{3/2}}\right]\Bigg\}$$

(could be simplified a little more).
The expressions inside the square brackets are positive, so F_x and F_y are negative. Thus the force is towards the conductor, i.e. attractive, which is as expected since the induced charge on the conductor planes (not calculated) should have opposite sign of q. Also, if $a=b$ the force should by symmetry point towards the corner $((x,y)=(0,0))$ which indeed it does:

$$a=b \Rightarrow \tan\frac{|F_y|}{|F_x|} = 1 \Rightarrow$$



Another interesting limit is $a \gg b$ (and $b \gg a$). When $a \gg b$ we would expect that the vertical plane is so far away that, to leading order, only the horizontal conducting plane should matter. Thus the problem reduces to the single-plane problem and the force should be downwards. This is confirmed by the expression for $\vec{F}_q$, which to leading order reduces to

$$\vec{F}_q = -\frac{q^2}{4\pi\epsilon_0} \frac{1}{(2b)^2} \hat{y}$$