TFY4240
Solution problem set 4

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Problem 1.
The electric field is

$$\boldsymbol{E}_{\text{dip}}(\boldsymbol{r}) = -\boldsymbol{\nabla} V_{\text{dip}}(\boldsymbol{r}) = -\frac{1}{4\pi\varepsilon_0}\boldsymbol{\nabla}\left(\frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3}\right) = -\frac{1}{4\pi\varepsilon_0} p_j \boldsymbol{\nabla}\left(\frac{r_j}{r^3}\right) \tag{1}$$

(we use the Einstein summation convention). We will need

$$\partial_i r = \partial_i \sqrt{r_k r_k} = \frac{1}{2r}\partial_i(r_k r_k) = \frac{1}{2r}\cdot 2r_k \underbrace{\partial_i r_k}_{\delta_{ik}} = \frac{r_i}{r}, \tag{2}$$

and (for the i'th component of Eq. (1))

$$\begin{aligned} p_j \partial_i\left(\frac{r_j}{r^3}\right) &= p_j\left[\frac{\partial_i r_j}{r^3} - \frac{r_j \partial_i r^3}{r^6}\right] \\ &= p_j\left[\frac{\delta_{ij}}{r^3} - \frac{r_j\, 3r^2 \partial_i r}{r^6}\right] \\ &= p_j\left[\frac{\delta_{ij}}{r^3} - 3\frac{r_j r_i}{r^5}\right] \\ &= \frac{p_i}{r^3} - 3\frac{(\boldsymbol{p}\cdot\boldsymbol{r})r_i}{r^5}. \end{aligned} \tag{3}$$

Substituting this expression back into Eq. (1) and introducing the unit vector $\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r}$ gives

$$\boldsymbol{E}_{\text{dip}}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{3(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p}}{r^3}. \tag{4}$$

Problem 2

(a) Since the potential on the sphere surface is independent of the azimuthal angle φ, the problem has azimuthal symmetry. We can therefore expand the potential as

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Since $V(r,\theta)$ should $\to 0$ as $r \to \infty$, A_l must be 0 for all l (including $l=0$)

$$\Rightarrow \quad V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \qquad (*)$$

To find the coefficients B_l, we need to consider the expansion $(*)$ for $r=R$ and use that it should equal $V(R,\theta) = V_0 \cos^2\theta$. The coefficients could then be determined by using the orthogonality relations for Legendre polynomials. However, because in this example $V(R,\theta) = V_0 \cos^2\theta$ has a simple expression in terms of Legendre polynomials, we can use that to simply read off the coefficient. To this end, we note that

$$P_0(\cos\theta) = 1 \quad , \quad P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$

$$\Rightarrow \quad \cos^2\theta - \frac{2}{3} P_2(\cos\theta) = \frac{1}{3} = \frac{1}{3} P_0(\cos\theta)$$

$$\Rightarrow \quad V(R,\theta) = \frac{V_0}{3}\left[P_0(\cos\theta) + 2\, P_2(\cos\theta) \right]$$

Comparing this with (*) evaluated at $r=R$, we can read off

$l=0$ term: $\frac{B_0}{R} = \frac{V_0}{3} \Rightarrow B_0 = \frac{V_0 R}{3}$

$l=2$ term: $\frac{B_2}{R^3} = \frac{2V_0}{3} \Rightarrow B_2 = \frac{2V_0 R^3}{3}$

$l \neq 0,2$: $\frac{B_l}{R^{l+1}} = 0 \Rightarrow B_l = 0$

Inserting these results back into (*) gives

$$V(r,\theta) = \frac{V_0}{3}\left[\frac{R}{r}\underbrace{P_0(\cos\theta)}_{=1} + 2\left(\frac{R}{r}\right)^3 P_2(\cos\theta)\right]$$

Hence the potential outside the sphere ($r \geq R$) is

$$\underline{\underline{V_{outside}(r,\theta) = \frac{V_0}{3}\left[\frac{R}{r} + 2\left(\frac{R}{r}\right)^3 P_2(\cos\theta)\right]}}$$

Let us check that this expression reduces to the correct result at $r=R$:

$$V_{outside}(R,\theta) = \frac{V_0}{3}\left[1 + 2P_2(\cos\theta)\right]$$
$$= \frac{V_0}{3}\left[1 + 2\cdot\frac{1}{2}(3\cos^2\theta - 1)\right] = \underline{V_0\cos^2\theta} \quad \text{ok}$$

(b) The azimuthal symmetry again allows us to write

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

B_ℓ must be 0 for all ℓ; otherwise $V(r,\theta)$ would diverge as $r \to 0$

$$\Rightarrow \quad V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell \, r^\ell \, P_\ell(\cos\theta) \qquad (**)$$

Evaluating this at $r=R$ and comparing with $V(R,\theta) = \frac{V_0}{3}\left[P_0(\cos\theta) + 2P_2(\cos\theta)\right]$ giv

$\ell=0$ term: $A_0 R^0 = \frac{V_0}{3} \Rightarrow A_0 = \frac{V_0}{3}$

$\ell=2$ term: $A_2 R^2 = \frac{V_0}{3}\cdot 2 \Rightarrow A_2 = \frac{2V_0}{3R^2}$

$\ell \neq 0,2$: $A_\ell R^\ell = 0 \Rightarrow A_\ell = 0$

Inserting these results back into $(**)$ gives

$$\underline{\underline{V_{inside}(r,\theta) = \frac{V_0}{3}\left[P_0(\cos\theta) + 2\left(\frac{r}{R}\right)^2 P_2(\cos\theta)\right]}}$$

(One can see that it reduces to $V_0\cos^2\theta$ at $r=R$, as it should.)

To summarize, we have found

$$V_{outside}(r,\theta) = \frac{V_0}{3}\left[\frac{R}{r} + 2\left(\frac{R}{r}\right)^3 P_2(\cos\theta)\right]$$

$$V_{inside}(r,\theta) = \frac{V_0}{3}\left[1 + 2\left(\frac{r}{R}\right)^2 P_2(\cos\theta)\right]$$

(c) The surface charge density σ on the spherical surface $r=R$ is given by

$$\sigma = -\epsilon_0\left[\left.\frac{\partial V_{outside}}{\partial n}\right|_{r=R} - \left.\frac{\partial V_{inside}}{\partial n}\right|_{r=R}\right]$$

On the spherical surface, $\hat{n} = \hat{r}$, so the normal derivative is

expression for ∇ in spherical coordinates

$$\frac{\partial}{\partial n} \equiv \hat{n}\cdot\nabla = \hat{r}\cdot\nabla$$

$$= \hat{r}\cdot\left[\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$= \frac{\partial}{\partial r} \qquad (\text{since } \hat{r}\cdot\hat{\theta} = \hat{r}\cdot\hat{\varphi} = 0)$$

$$\Rightarrow \sigma = -\epsilon_0\left[\left.\frac{\partial V_{outside}}{\partial r}\right|_{r=R} - \left.\frac{\partial V_{inside}}{\partial r}\right|_{r=R}\right]$$

We have

$$\frac{\partial V_{outside}}{\partial r} = \frac{V_0}{3}\left[-\frac{R}{r^2} + 2\cdot 3\left(\frac{R}{r}\right)^2\cdot\left(\frac{-R}{r^2}\right)P_2(\cos\theta)\right]$$

$$\Rightarrow \left.\frac{\partial V_{outside}}{\partial r}\right|_{r=R} = -\frac{V_0}{3}\left[1 + 6P_2(\cos\theta)\right]$$

$$\frac{\partial V_{inside}}{\partial r} = \frac{V_0}{3}\cdot 2\cdot 2\frac{r}{R}\cdot\frac{1}{R}P_2(\cos\theta)$$

$$\Rightarrow \left.\frac{\partial V_{inside}}{\partial r}\right|_{r=R} = \frac{4V_0}{3R}P_2(\cos\theta)$$

The surface charge density is therefore

$$\sigma = -\epsilon_0\frac{V_0}{3R}\left\{-\left(1+6P_2(\cos\theta)\right) - 4P_2(\cos\theta)\right\}$$

$$= \underline{\underline{\frac{\epsilon_0 V_0}{3R}\left[1 + 10P_2(\cos\theta)\right] \equiv \sigma(\theta)}}$$

Problem 3

(a) As the surface charge density only depends on θ, not the azimuthal angle φ, the resulting potential will also have azimuthal symmetry, i.e. $V = V(r,\theta)$, so it can be expanded as

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

There are really two such expansions, one for $r \geq R$ (outside the sphere) and one for $r \leq R$ (inside the sphere). Each expansion has its own set of coefficients $\{A_l, B_l\}$. The two expansions must be matched at $r = R$ using the two boundary/matching conditions

BC 1: $\quad V_{outside}(R,\theta) = V_{inside}(R,\theta)$

BC 2: $\quad \partial_n V_{outside}(r,\theta)\Big|_{r=R} - \partial_n V_{inside}(r,\theta)\Big|_{r=R} = -\dfrac{\sigma(\theta)}{\epsilon_0}$

First consider $V_{outside}$. Infinitely far away from the charge, $V_{outside}$ must go to 0, i.e.

$$V_{outside}(r,\theta) \to 0 \text{ as } r \to \infty \Rightarrow A_l^{(outside)} = 0 \text{ for all } l$$

$$\Rightarrow \quad V_{outside}(r,\theta) = \sum_l \frac{B_l^{(outside)}}{r^{l+1}} P_l(\cos\theta)$$

Next consider V_{inside}. To avoid V_{inside} diverging as $r \to 0$, $B_\ell^{(inside)}$ must be 0 for all ℓ. Thus

$$V_{inside}(r,\theta) = \sum_\ell A_\ell^{(inside)} r^\ell P_\ell(\cos\theta)$$

At this point I will rename $B_\ell^{(outside)} \equiv B_\ell$ and $A_\ell^{(inside)} \equiv A_\ell$ to save some writing

BC 1 gives

$$\sum_\ell \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) = \sum_\ell A_\ell R^\ell P_\ell(\cos\theta)$$

Focusing only on the θ dependence now, this equality can be written

$$\sum_\ell f_\ell P_\ell(\cos\theta) = \sum_\ell g_\ell P_\ell(\cos\theta) \qquad (*)$$

For such an equality to hold, the coefficients of the two series must be equal, i.e.

$$f_\ell = g_\ell \qquad \text{for all } \ell$$

(To see this, multiply $(*)$ by $P_{\ell'}(\cos\theta)\sin\theta$, integrate over θ from 0 to π, and use the orthogonality of the Legendre polynomials:

$$0 = \sum_\ell [f_\ell - g_\ell] \underbrace{\int_0^\pi d\theta \sin\theta \, P_\ell(\cos\theta) P_{\ell'}(\cos\theta)}_{\int_{-1}^{1} dx \, P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell\ell'}}$$

$$\Rightarrow 0 = (f_{\ell'} - g_{\ell'}) \cdot \frac{2}{2\ell'+1} \Rightarrow f_{\ell'} = g_{\ell'} \;)$$

Equating coefficients in BC 1 gives

$$\frac{B_\ell}{R^{\ell+1}} = A_\ell R^\ell \Rightarrow \underline{B_\ell = R^{2\ell+1} A_\ell} \quad (**)$$

Next consider BC 2. Since the surface is spherical, $\partial_n = \partial_r$. We find

$$\partial_r V_{outside}(r,\theta) = \sum_\ell B_\ell \left[-(\ell+1)\right] \frac{1}{r^{\ell+2}} P_\ell(\cos\theta)$$

$$\partial_r V_{inside}(r,\theta) = \sum_\ell A_\ell \, \ell \, r^{\ell-1} P_\ell(\cos\theta)$$

Also, $\sigma(\theta) = \sum_\ell \sigma_\ell P_\ell(\cos\theta)$. Thus BC2 gives

$$\sum_\ell \left[- \frac{B_\ell(\ell+1)}{R^{\ell+2}} - A_\ell \, \ell \, R^{\ell-1} \right] P_\ell(\cos\theta) = -\frac{1}{\epsilon_0} \sum_\ell \sigma_\ell P_\ell(\cos\theta)$$

This is again of the form (*). Equating coefficients gives

$$-\frac{B_\ell(\ell+1)}{R^{\ell+2}} - A_\ell \, \ell \, R^{\ell-1} = -\frac{\sigma_\ell}{\epsilon_0}$$

Inserting (**) gives

$$\left[-(\ell+1) - \ell\right] A_\ell R^{\ell-1} = -\frac{\sigma_\ell}{\epsilon_0}$$

$$\Rightarrow \quad A_\ell = \frac{\sigma_\ell}{\epsilon_0 (2\ell+1) R^{\ell-1}}$$

$$B_\ell = \frac{\sigma_\ell R^{\ell+2}}{\epsilon_0 (2\ell+1)}$$

which gives

$$V_{\text{outside}}(r,\theta) = \frac{R}{\epsilon_0} \sum_{l=0}^{\infty} \frac{\sigma_l}{2l+1} \left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta)$$

$$V_{\text{inside}}(r,\theta) = \frac{R}{\epsilon_0} \sum_{l=0}^{\infty} \frac{\sigma_l}{2l+1} \left(\frac{r}{R}\right)^{l} P_l(\cos\theta)$$

As a check, we see that the expressions agree on the spherical surface $r=R$, as they should:

$$V(R,\theta) = \frac{R}{\epsilon_0} \sum_{l=0}^{\infty} \frac{\sigma_l}{2l+1} P_l(\cos\theta)$$

(b) The coefficients σ_l for $\sigma(\theta)$ in 2c can be read off as (using that $P_0(\cos\theta)=1$)

$$\sigma_l = \frac{\epsilon_0 V_0}{3R} \left(\delta_{l,0} + 10\, \delta_{l,2} \right)$$

This gives

$$V_{\text{outside}}(r,\theta) = \frac{\epsilon_0 V_0}{3R} \cdot \frac{R}{\epsilon_0} \sum_{l=0}^{\infty} \frac{\delta_{l,0} + 10\,\delta_{l,2}}{2l+1} \left(\frac{R}{r}\right)^{l+1} P_l(\cos\theta)$$

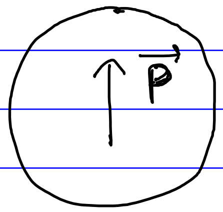
$$= \frac{V_0}{3} \left[\frac{1}{2\cdot 0+1} \left(\frac{R}{r}\right)^{0+1} P_0(\cos\theta) + \frac{10}{2\cdot 2+1} \left(\frac{R}{r}\right)^{2+1} P_2(\cos\theta) \right]$$

$$= \frac{V_0}{3} \left[\frac{R}{r} + 2 \left(\frac{R}{r}\right)^3 P_2(\cos\theta) \right],$$

$$V_{\text{inside}}(r,\theta) = \frac{\epsilon_0 V_0}{3R} \cdot \frac{R}{\epsilon_0} \sum_{l=0}^{\infty} \frac{\delta_{l,0} + 10\,\delta_{l,2}}{2l+1} \left(\frac{r}{R}\right)^{l} P_l(\cos\theta)$$

$$= \frac{V_0}{3} \left[1 + 2 \left(\frac{r}{R}\right)^2 P_2(\cos\theta) \right],$$ as found in 2a & 2b

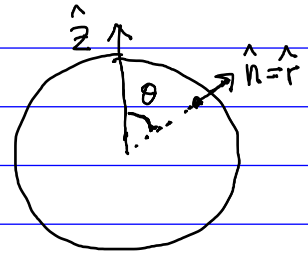
(c) Dielectric sphere with uniform polarization $\vec{P}$:



Choose the z-axis to be parallel to $\vec{P}$, i.e. $\vec{P} = P\hat{z}$

Since $\vec{P}$ is constant, the <u>volume</u> surface bound charge density $\rho_b = -\nabla\cdot\vec{P} = 0$

<u>Surface</u> bound charge density:



$$\sigma_b = \vec{P}\cdot\hat{n} = P\,\hat{z}\cdot\hat{r} = P\cos\theta$$

$$= P\,P_1(\cos\theta)$$

This is therefore a problem involving an azimuthally symmetric surface charge density on a sphere, so the method in 3a can be applied to find the potential $V^{(P)}$ due to the polarization

The coefficients $\{\sigma_\ell\}$ can be read off as

$$\sigma_\ell = P\,\delta_{\ell,1}$$

Thus

$$V^{(P)}_{\text{outside}}(r,\theta) = \frac{R}{\epsilon_0}\,\frac{P}{2\cdot 1+1}\left(\frac{R}{r}\right)^{1+1} P_1(\cos\theta)$$

$$= \frac{PR^3\cos\theta}{3\epsilon_0 r^2}$$

The total dipole moment $\vec{p}$ of the sphere is

$$\vec{p} = \int d^3r\,\vec{P}(\vec{r}) = \vec{P}\cdot\frac{4}{3}\pi R^3$$

$$\Rightarrow \quad V^{(P)}_{\text{outside}}(r,\theta) = \frac{1}{4\pi\epsilon_0}\,\frac{p\cos\theta}{r^2} = \frac{1}{4\pi\epsilon_0}\,\frac{\vec{p}\cdot\hat{r}}{r^2}$$

This is the potential of a pure/point dipole (with electric dipole moment $\vec{p}$, located at the origin)

Thus the electric field $\vec{E}^{(P)}_{\text{outside}}(\vec{r})$ is the field of a pure/point electric dipole.

Next, we consider $V^{(P)}$ inside the sphere:

$$V^{(P)}_{\text{inside}}(r,\theta) = \frac{R}{\epsilon_0}\,\frac{P}{2\cdot 1+1}\left(\frac{r}{R}\right)^1 P_1(\cos\theta) = \frac{P}{3\epsilon_0}\,z$$

since $z = r\cos\theta$.

Since $V^{(P)}_{\text{inside}}$ can be expressed as a function of z only, it is convenient to calculate the electric field using cartesian coordinates:

$$\vec{E}^{(P)}_{\text{inside}}(\vec{r}) = -\nabla V^{(P)}_{\text{inside}}(\vec{r})$$

$$= -\frac{P}{3\epsilon_0}\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) z$$

$$= -\frac{P}{3\epsilon_0}\left(\hat{x}\cdot 0 + \hat{y}\cdot 0 + \hat{z}\cdot 1\right) = -\frac{P\hat{z}}{3\epsilon_0} = -\frac{\vec{P}}{3\epsilon_0}$$

So inside the sphere, the field due to the polarization is <u>constant</u>, pointing in the direction opposite to $\vec{P}$. It therefore points from the region of positive bound surface charge to the region of negative bound surface charge, as expected ($\sigma_b = P\cos\theta$ has opposite signs for $0 \le \theta < \pi/2$ and $\pi/2 < \theta \le \pi$).