TFY4240
Solution problem set 5

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Problem 1.
See handwritten solution further back.

Problem 2.
See Griffiths.

Problem 3.

a) The magnetic field from a line element $d\boldsymbol{l}'$ is given by Biot-Savart's law,

$$d\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \frac{d\boldsymbol{l}' \times \hat{\boldsymbol{R}}}{R^2} \tag{1}$$

where $d\boldsymbol{l}'$ points along the wire, in the direction of the current, and $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$. At position O (the origin), $\boldsymbol{r} = \boldsymbol{0}$, so $\boldsymbol{R} = -\boldsymbol{r}'$, giving (using $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$)

$$d\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \frac{\hat{\boldsymbol{r}}' \times d\boldsymbol{l}'}{r'^2}. \tag{2}$$

There is no contribution to the magnetic field from the straight parts of the wire, since the line elements $d\boldsymbol{l}'$ along those parts are parallel to $\hat{\boldsymbol{r}}'$, so $\hat{\boldsymbol{r}}' \times d\boldsymbol{l}' = 0$. Thus the only contribution comes from the semicircle. There $\hat{\boldsymbol{r}}' \times d\boldsymbol{l}'$ points into the paper plane, so therefore the magnetic field does too. Using also that along the semicircle (i) $d\boldsymbol{l}'$ is perpendicular to $\hat{\boldsymbol{r}}'$, so $|\hat{\boldsymbol{r}}' \times d\boldsymbol{l}'| = dl'$, (ii) $r' = R$, the magnitude of the field becomes (the integral goes over the semicircle only)

$$B = \int dB = \frac{\mu_0 I}{4\pi R^2} \underbrace{\int dl'}_{\pi R} = \frac{\mu_0 I}{4R} \tag{3}$$

since $\int dl'$ is just the semicircle length. As already noted, $\boldsymbol{B}$ points into the paper plane.

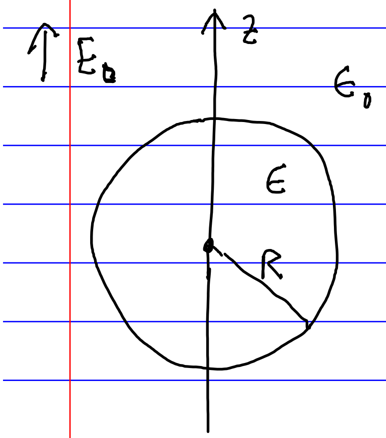
b)

$$|\boldsymbol{B}| = \frac{\mu_0 I}{4R} = 4\pi \cdot 10^{-7}\ \mathrm{N/A^2} \cdot \frac{1}{4} \cdot 10^2\ \mathrm{A/m} \approx 3 \cdot 10^{-5}\ \mathrm{T}. \tag{4}$$

Problem 4.
See handwritten solution further back.

Problem 5.
See handwritten solution further back.

Problem 1 Poisson equation : $\nabla^2 V = -\frac{\rho_f(\vec{r})}{\epsilon_m}$



where ϵ_m is the permittivity of the medium (here: $\epsilon_m = \epsilon$ inside, $\epsilon_m = \epsilon_0$ outside)

Boundary conditions at $r = R$:

(1) $V_{outside} = V_{inside}$

(2) $\epsilon_{outside}\, \partial_n V_{outside} - \epsilon_{inside}\, \partial_n V_{inside} = -\sigma_f$

In this problem there is no free charge $\Rightarrow \rho_f, \sigma_f = 0$

$\Rightarrow$ Both inside and outside the sphere the Poisson eq reduces to the Laplace equation

$$\nabla^2 V = 0$$

and BC (2) reduces to $\epsilon_0 \partial_r V_{outside} = \epsilon\, \partial_r V_{inside}$

Let $\vec{E}_0 = E_0 \hat{z}$ and let the z axis pass through the center of the sphere. The problem has azimuthal symmetry, so the solution of the Laplace eq. can be expanded as

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \qquad (*)$$

(with different expansions inside and outside the sphere)

For $V_{outside}(r,\theta)$ we must consider the boundary condition as $r \to \infty$. Far away from the sphere, $\vec{E}$ must approach the applied field $\vec{E}_0$:

$$\vec{E}_{outside} = -\nabla V_{outside} \to E_0 \hat{z} \quad \text{as} \quad r \to \infty$$

$$\Rightarrow \quad V_{outside}(r,\theta) \to -E_0 z + C \quad \text{as} \quad r \to \infty$$

We pick the constant $C = 0$ so that in the absence of E_0, the boundary condition reduces to $V \to 0$. As $z = r\cos\theta$, the boundary condition thus becomes

$$V_{outside}(r,\theta) \to -E_0 r\cos\theta \qquad \text{as} \quad r \to \infty$$

Comparing this with $(*)$ it follows that

$$A_1^{(outside)} = -E_0$$
$$A_\ell^{(outside)} = 0 \quad \text{for } \ell \neq 1$$

Since the terms $\propto B_\ell^{(outside)}$ all go to 0 as $r \to \infty$, we do not get any conditions on these B-coefficients from this BC. Thus $V_{outside}$ reduces to

$$V_{outside}(r,\theta) = -E_0 r\cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell^{(outside)}}{r^{\ell+1}} P_\ell(\cos\theta)$$

For $V_{inside}(r,\theta)$, $B_\ell^{(inside)}$ must be 0 for all ℓ to prevent V_{inside} from diverging as $r \to 0$. Thus V_{inside} reduces to

$$V_{inside}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell^{(inside)} r^\ell P_\ell(\cos\theta)$$

Now I rename $A_\ell^{(inside)} \equiv A_\ell$, $B_\ell^{(outside)} \equiv B_\ell$

BC (1) gives

$$-E_0 R\cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta)$$

Equating coefficients of P_ℓ for each ℓ gives (note that $\cos\theta = P_1(\cos\theta)$)

$$\ell = 1: \quad -E_0 R + \frac{B_1}{R^2} = A_1 R \Rightarrow B_1 = R^3 (A_1 + E_0)$$

$$\ell \neq 1: \quad \frac{B_\ell}{R^{\ell+1}} = A_\ell R^\ell \Rightarrow B_\ell = R^{2\ell+1} A_\ell$$

Next we consider BC (2). We have

$$\partial_r V_{\text{outside}}(r,\theta) = -E_0 \cos\theta + \sum_{\ell=0}^{\infty} (-1)(\ell+1) \frac{B_\ell}{r^{\ell+2}} P_\ell(\cos\theta)$$

$$\partial_r V_{\text{inside}}(r,\theta) = \sum_{\ell=0}^{\infty} \ell A_\ell r^{\ell-1} P_\ell(\cos\theta)$$

Thus BC (2) becomes

$$\epsilon_0 \left[-E_0 P_1(\cos\theta) - \sum_{\ell=0}^{\infty} (\ell+1) \frac{B_\ell}{R^{\ell+2}} P_\ell(\cos\theta) \right] = \epsilon \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos\theta)$$

Equating coefficients for each ℓ gives

$$\ell = 1: \quad \epsilon_0 \left[-E_0 - \frac{2B_1}{R^3} \right] = \epsilon A_1$$

$$\ell \neq 1: \quad -\epsilon_0 (\ell+1) \frac{B_\ell}{R^{\ell+2}} = \epsilon \ell A_\ell R^{\ell-1}$$

Inserting the relation between B_ℓ and A_ℓ obtained from BC (1) gives

$$\ell = 1: \quad \epsilon_0 \left[-E_0 - \frac{2}{R^3} R^3 (A_1 + E_0) \right] = \epsilon A_1$$

$$\Rightarrow: \quad \epsilon_0 [-2A_1 - 3E_0] = \epsilon A_1 \Rightarrow A_1 = -\frac{3\epsilon_0}{2\epsilon_0 + \epsilon} E_0 = -\frac{3}{K+2} E_0$$

$$\Rightarrow \quad B_1 = R^3\left[-\frac{3}{2+K} + 1\right]E_0 = \frac{K-1}{K+2}E_0 R^3$$

$$l \neq 1: \; -\epsilon_0 (l+1)\frac{R^{2l+1}}{R^{l+2}} A_l = \epsilon\, l\, A_l\, R^{l-1} \Rightarrow A_l = 0 \Rightarrow B_l = 0$$

Thus $V_{outside}(r,\theta) = -E_0 r\cos\theta + \frac{K-1}{K+2}E_0 R^3 \frac{1}{r^2}\cos\theta$

The first term is due to the applied field $\vec{E}_0$.
The second term is the dipole potential

$$V_{dip}(r,\theta) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} \quad \text{with} \quad p = 4\pi\epsilon_0 \frac{K-1}{K+2}R^3 E_0$$

being the dipole moment. Thus $\vec{E}_{outside}$ is the sum of $\vec{E}_0$ and the polarization field $\vec{E}_p$ which is a dipole field. Furthermore,

$$V_{inside}(r,\theta) = -\frac{3}{K+2}E_0\, r\cos\theta = -\frac{3}{K+2}E_0\, z$$

Thus $\underline{\underline{\vec{E}_{inside}}} = -\nabla V_{inside} = \frac{3}{K+2}E_0\hat{z} = \underline{\underline{\frac{3}{K+2}\vec{E}_0}}$

As $K > 1$ we see that $|\vec{E}_{inside}| < E_0$, i.e. the applied field is partially screened by the dielectric. Polarization:

$$\underline{\underline{\vec{P}(\vec{r})}} = \epsilon_0 \chi \vec{E}_{inside}(\vec{r}) = \underline{\underline{3\epsilon_0 E_0 \frac{K-1}{K+2}\hat{z}}} \quad \text{(a constant)}$$

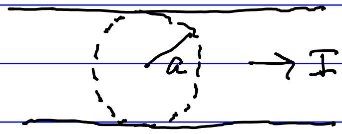
(and $\vec{P} = 0$ everywhere outside the sphere). The dipole moment is

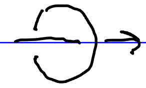
$$\vec{p} \equiv \int_{sphere} d^3r\, \vec{P}(\vec{r}) = \vec{P}\,\frac{4\pi}{3}R^3 = \underline{\underline{4\pi\epsilon_0 \frac{K-1}{K+2}R^3\vec{E}_0}}$$

(agrees with p found from V_{dip} above)

Remark: The limit $K \to \infty$ of these results reproduces the results for a perfectly conducting sphere in an external field: $\vec{E}_{inside} = 0$ (i.e. complete screening) and $\vec{p} = 4\pi\epsilon_0 R^3 \vec{E}_0$

Problem 4



Consider a circle of radius r about the cylinder axis. $\vec{B}$ will have the same magnitude at all points on the circle and be oriented tangentially to the circle with a direction given by a right-hand rule: 

Because of the symmetry it is useful to apply Ampere's law on integral form, picking the Amperian loop C to coincide with the circumference of the circle of radius r:

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enclosed} \Rightarrow B \cdot 2\pi r = \mu_0 I_{encl}$$

$$\Rightarrow \underline{B = \frac{\mu_0 I_{encl}}{2\pi r}}$$

(a) $r < a \Rightarrow I_{encl} = 0 \Rightarrow \underline{\underline{B = 0}}$

$r > a \Rightarrow I_{encl} = I \Rightarrow \underline{\underline{B = \frac{\mu_0 I}{2\pi r}}}$

(b) $j = ks$ for some constant k

The value of k can be found from the condition that the total current is known to be I

$$\Rightarrow I = \int_a j\, da = \int_0^{2\pi} d\varphi \int_0^a ds\, s\, j = k \cdot 2\pi \int_0^a ds\, s^2$$

$$= 2\pi k \frac{1}{3} a^3 \Rightarrow \underline{k = \frac{3I}{2\pi a^3}}$$

For $r < a$,

$$I_{encl} = \int_0^{2\pi} d\varphi \int_0^r ds\, s\, ks = 2\pi \frac{3I}{2\pi a^3} \frac{1}{3} r^3 = I\left(\frac{r}{a}\right)^3$$

$$\Rightarrow \quad B = \frac{\mu_0 I_{encl}}{2\pi r} = \underline{\underline{\frac{\mu_0 I r^2}{2\pi a^3}}}$$

For $r > a$, $I_{encl} = I \Rightarrow \underline{B = \frac{\mu_0 I}{2\pi r}}$

Problem 5

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\vec{j}(\vec{r}') \times \hat{R}}{R^2} \qquad (\vec{R} = \vec{r} - \vec{r}')$$

Note: ∇ acts on $\vec{r}$ (as always)

(a) $$\nabla \cdot \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \nabla \cdot \int d^3r' \frac{\vec{j}(\vec{r}') \times \hat{R}}{R^2}$$

$$= \frac{\mu_0}{4\pi} \int d^3r' \, \nabla \cdot \left(\vec{j}(\vec{r}') \times \frac{\hat{R}}{R^2}\right)$$

Use $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$ (vector identity (6) in Griffiths)

$$\Rightarrow \nabla \cdot \left(\vec{j}(\vec{r}') \times \frac{\hat{R}}{R^2}\right) = \frac{\hat{R}}{R^2} \cdot \underbrace{\left(\nabla \times \vec{j}(\vec{r}')\right)}_{\substack{=0 \text{ since} \\ \vec{j} \text{ is a function} \\ \text{of } \vec{r}', \text{ not } \vec{r}}} - \vec{j}(\vec{r}') \cdot \underbrace{\left(\nabla \times \frac{\hat{R}}{R^2}\right)}_{\substack{=0 \text{ (we showed} \\ \text{this in Problem} \\ \text{3 in Tutorial 2)}}}$$

$$\Rightarrow \quad \underline{\underline{\nabla \cdot \vec{B}(\vec{r}) = 0}}$$

(b) $\nabla \times \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \nabla \times \left(\vec{j}(\vec{r}') \times \frac{\hat{R}}{R^2} \right)$

The curl inside the integral takes the form

$\nabla \times (\vec{a} \times \vec{b})$ with $\vec{a} = \vec{j}(\vec{r}')$ and $\vec{b} = \frac{\hat{R}}{R^2}$

We use the identity (identity (8) in Griffiths)

$$\nabla \times (\vec{a} \times \vec{b}) = (\vec{b} \cdot \nabla)\vec{a} - (\vec{a} \cdot \nabla)\vec{b} + \vec{a}(\nabla \cdot \vec{b}) - \vec{b}(\nabla \cdot \vec{a})$$

The fact that $\vec{a} = \vec{j}(\vec{r}')$ does not depend on $\vec{r}$ implies that the first and last terms on the rhs vanish, giving

$$\nabla \times \left(\vec{j}(\vec{r}') \times \frac{\hat{R}}{R^2} \right) = -\left(\vec{j}(\vec{r}') \cdot \nabla \right) \frac{\hat{R}}{R^2} + \vec{j}(\vec{r}') \left(\nabla \cdot \frac{\hat{R}}{R^2} \right) \quad (*)$$

Consider the first term on the rhs:

$$-(\vec{j} \cdot \nabla) \frac{\hat{R}}{R^2} = -j_i \partial_i \frac{\hat{R}}{R^2} = j_i \partial_i' \frac{\hat{R}}{R^2} \quad (**)$$

where $\partial_i' \equiv \frac{\partial}{\partial r_i'}$. Proof: $\partial_i \frac{\hat{R}}{R^2} = \partial_i \frac{\vec{R}}{R^3}$ (a vector)

Consider an arbitrary component k: $\partial_i \frac{R_k}{R^3} = \frac{\partial}{\partial r_i} \frac{R_k}{R^3}$

$= \frac{\partial R_k}{\partial r_i} \frac{\partial}{\partial R_k} \frac{R_k}{R^3}$ (to be clear: no sum over k here)

Here, $\frac{\partial R_k}{\partial r_i} = \frac{\partial (r_k - r_k')}{\partial r_i} = \delta_{ik} = -\frac{\partial (r_k - r_k')}{\partial r_i'} = -\frac{\partial R_k}{\partial r_i'}$

So differentiating R_k with respect to r_i' instead of r_i gives a relative minus sign

$\Rightarrow \partial_i \frac{R_k}{R^3} = -\partial_i' \frac{R_k}{R^3}$, from which $(**)$ follows

$$j_i \, \partial_i' \frac{\hat{R}}{R^2} = (\vec{j}\cdot\nabla')\frac{\hat{R}}{R^2} = (\vec{j}\cdot\nabla')\frac{\vec{R}}{R^3}$$

Consider the k'th component : $(\vec{j}\cdot\nabla')\frac{R_k}{R^3}$

Use the identity (identity (5) in Griffiths)

$$\nabla\cdot(f\vec{a}) = f\,\nabla\cdot\vec{a} - \vec{a}\cdot\nabla f$$

$$\Rightarrow \quad \vec{a}\cdot\nabla f = f\,\nabla\cdot\vec{a} - \nabla\cdot(f\vec{a})$$

Take $\vec{a} = \vec{j}$, $f = \frac{R_k}{R^3}$, $\nabla \to \nabla'$ to get

$$\vec{j}\cdot\nabla'\frac{R_k}{R^3} = \frac{R_k}{R^3}\underbrace{\nabla'\cdot\vec{j}(\vec{r}')}_{=0} - \nabla'\cdot\left(\frac{R_k}{R^3}\vec{j}\right) = -\nabla'\cdot\left(\frac{R_k}{R^3}\vec{j}\right) \quad (***)$$

where we used the steady-current condition $\nabla'\cdot\vec{j}(\vec{r}') = 0$ valid for static problems. Now put (***) back into the integral. This integral can be done using the divergence theorem. This gives (omitting constant factors)

$$\int d^3r' \, \nabla'\cdot\left(\frac{R_k}{R^3}\vec{j}(\vec{r}')\right) = \int d\vec{a}'\cdot\left(\frac{R_k}{R^3}\vec{j}(\vec{r}')\right) = 0 .$$

The volume integral is over all space (or at least over any region big enough to completely contain the current in its interior; we assume a localized current distribution). Therefore $\vec{j}(\vec{r}') = 0$ in the surface integral, which therefore vanishes.

It remains to consider the 2nd term on the rhs of (*). Putting it back in the integral gives

$$\underline{\underline{\nabla\times\vec{B}(\vec{r})}} = \frac{\mu_0}{4\pi}\int d^3r' \, \vec{j}(\vec{r}')\underbrace{\left(\nabla\cdot\frac{\hat{R}}{R^2}\right)}_{=4\pi\delta(\vec{R})} = \underline{\underline{\mu_0\,\vec{j}(\vec{r})}} \quad \text{(phew!)}$$