TFY4240
Problem set 6

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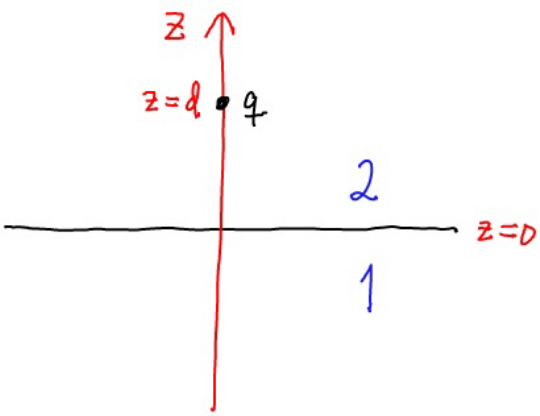
Problem 1.



Let space be divided into two regions filled with simple dielectric media 1 and 2 (with dielectric constants κ_1 and κ_2), respectively. The interface between the regions is an infinite plane. A point charge q sits in medium 2 a distance d from the interface.

We introduce a cartesian coordinate system with the z axis perpendicular to the interface and passing through the point charge, with origin and direction chosen such that the interface is at $z = 0$, medium 1 is in the region $z < 0$, medium 2 is in the region $z > 0$, and the point charge is at $z = d$ (see figure).

a) Find the potential $V(\boldsymbol{r})$ at an arbitrary point $\boldsymbol{r}$. Hints: Define

$$V(\boldsymbol{r}) = \begin{cases} V_1(\boldsymbol{r}) & \text{for } z < 0, \\ V_2(\boldsymbol{r}) & \text{for } z > 0. \end{cases} \tag{1}$$

The two functions $V_1(\boldsymbol{r})$ and $V_2(\boldsymbol{r})$ can be found using the method of images. This will be a more advanced example of the use of this method than you have encountered before. Do as follows: To find $V_1(\boldsymbol{r})$, remove medium 2 (and the point charge q embedded in it) and let all space be filled with medium 1. Put an image charge q_1 on the z axis at $z = d$ and calculate the potential due to this image charge. To find $V_2(\boldsymbol{r})$, remove medium 1 and let all space be filled with medium 2. Put an image charge q_2 on the z axis at $z = -d$ and calculate the potential due to this image charge and the point charge q. (In both cases, note that we do not change anything in the region where the potential is sought.) Next, show that the two boundary conditions at the interface (one for V and one for its derivative) can now be satisfied if the image charges are

$$q_1 = \frac{2\kappa_1}{\kappa_1 + \kappa_2} q, \tag{2}$$

$$q_2 = -\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} q. \tag{3}$$

b) Find the force $\boldsymbol{F}$ on the point charge q.

Problem 2.
The vector potential in the Coulomb gauge reads

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\boldsymbol{j}(\boldsymbol{r}')}{R} \tag{4}$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$ and $R = |\boldsymbol{R}|$.

a) Show that the expression (4) satisfies the Coulomb gauge condition $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$.

b) Calculate $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$ and show that it can be written as the Biot-Savart law.

Problem 3.
Consider a magnetic point dipole with magnetic dipole moment $\boldsymbol{m}$. In a coordinate system centered at the dipole, the vector potential associated with the dipole can in the Coulomb gauge be written

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\hat{\boldsymbol{r}}}{r^2}. \tag{5}$$

a) Use expression (5) to obtain an expression for the magnetic field $\boldsymbol{B}(\boldsymbol{r})$. Express your answer in *coordinate-free form.*

b) Compare your answer from the previous subproblem with the expression for the electric field from an electric point dipole.

Problem 4.
A uniform external magnetic field $\boldsymbol{B}_0$ induces a uniform magnetization $\boldsymbol{M}$ inside a sphere of radius R made of a simple magnetizable medium with permeability μ.

a) Find the total magnetic field $\boldsymbol{B}$ inside the sphere. [Hint: You may use the results derived in the lectures for the magnetic field $\boldsymbol{B}_M$ due to a sphere with uniform magnetization $\boldsymbol{M}$.]

b) For $\boldsymbol{B}$ inside the sphere, what is the condition for having (i) $|\boldsymbol{B}| > |\boldsymbol{B}_0|$ (ii) $|\boldsymbol{B}| = |\boldsymbol{B}_0|$, (iii) $|\boldsymbol{B}| < |\boldsymbol{B}_0|$, (iv) $|\boldsymbol{B}| = 0$ (i.e. complete screening of the external magnetic field)? For each case, express the condition in terms of the magnetic permeability μ, the relative permeability $\kappa_m = \mu/\mu_0$, and the magnetic susceptibility χ_m.

c) Find the magnetization $\boldsymbol{M}$ and the magnetic moment $\boldsymbol{m}$ of the sphere.

d) Find $\boldsymbol{B}$ outside the sphere.

Problem 5.
Examples 6.1, 6.2, and 6.3 from Griffiths.