TFY4240

Solution problem set 7

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Problem 1.
See handwritten solution further back.

Problem 2.
See handwritten solution further back.

Problem 3.
See Griffiths.

Problem 4.

a) We will use the known result for the magnetic field due to an infinite straight current-carrying wire: The magnitude of the field is $\mu_0 I/2\pi s$, where s is the distance to the wire, and the direction of the field is "circumferential" (i.e. the field "winds around" the wire). Referring to the figure in the problem text, for a current I towards the right, the field points out of the paper above the wire (where the loop is).

The magnetic flux through the loop of wire is given by

$$\Phi = \int_a d\boldsymbol{a} \cdot \boldsymbol{B} \tag{1}$$

where a is any surface that is bounded by the loop of wire. (Can you show that the magnetic flux is the same for any two such surfaces? Hint: By noting that two such surfaces together form a *closed* surface, use Gauss's law for the magnetic field to show that the difference in flux through the two surfaces is zero.) Because of the form of the magnetic field, the flux is here most easily calculated by making the "most obvious choice" of surface, namely the flat square surface bounded by the loop.

Next, in order to fix the sign of the flux, we should choose whether the normal unit vector $\hat{\boldsymbol{n}}$ for this surface should point into or out of the paper (the physics is of course unaffected by this choice). Since $\boldsymbol{B}$ points out of the paper, we will choose $\hat{\boldsymbol{n}}$ out of the paper too, as that makes the flux positive. Thus

$$\boldsymbol{\Phi} = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a} = \frac{\mu_0 I}{2\pi} \int_{\ell}^{\ell+L} \frac{1}{s} L \mathrm{d}s = \frac{\mu_0 I L}{2\pi} \ln\left(\frac{\ell + L}{\ell}\right) = \frac{\mu_0 I L}{2\pi} \ln\left(1 + \frac{L}{\ell}\right). \tag{2}$$

b) In this situation the distance ℓ becomes time-dependent, with $d\ell/dt = v > 0$. Thus the flux Φ becomes time-dependent too. The emf ε that is generated can be calculated from the "flux rule":

$$\varepsilon = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mu_0 I L}{2\pi} \frac{1}{1 + L/\ell} L \left(-\frac{1}{\ell^2}\right) \frac{\mathrm{d}\ell}{\mathrm{d}t} = \frac{\mu_0 I L^2 v}{2\pi \ell(t)[\ell(t) + L]}. \tag{3}$$

The direction of the induced current is counterclockwise. This can be shown in several ways (in the following, the "external" field refers to the field produced by the infinite wire):

- From the sign of ε and the direction of $\hat{\boldsymbol{n}}$: Since $\hat{\boldsymbol{n}}$ was chosen out of the paper, the reference direction for traversing the loop is counterclockwise (this follows from a right-hand rule: curling the fingers of your right hand in the reference direction of traversal, your thumb points in the direction of $\hat{\boldsymbol{n}}$). Since our result (3) shows that the sign of ε is *positive*, the induced current is in the *same* direction as the the reference direction, i.e. counterclockwise.
- From Lenz's law: Since the magnitude of the flux through the loop gets weaker as the loop is moved away, the induced current will try to increase this magnitude by having the induced field point in the same direction as the "external" field inside the loop, i.e. out of the page. It follows from a right-hand rule that the induced current must then be counterclockwise.
- From the Lorentz force (since the emf is motional): The "external" field points out of the page, so the magnetic Lorentz force on a charge in the segment of the loop closest to the infinite wire is to the right. The force on a charge on the far side of the loop is also to the right, but here the field and therefore the force is weaker, and thus the direction set by the closest loop segment wins, giving a counterclockwise current.

c) Since the magnetic field only depends on the distance to the wire, moving the loop to the right, i.e. parallel to the wire, will not change the flux through the loop, so the emf $\varepsilon = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = 0$. Thus the induced current is also zero.

Problem 5.

a) To calculate $\boldsymbol{\nabla} \cdot \overleftrightarrow{T}$ one has to keep in mind that $\boldsymbol{\nabla} = \hat{\boldsymbol{x}}_i \partial_i$ operates on what appears to its right. The effect of $\boldsymbol{\nabla}$ on $\overleftrightarrow{T} = T_{ij} \hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j$ is limited to the components T_{ij} since the Cartesian unit vectors $\hat{\boldsymbol{x}}_i$ are independent of $\boldsymbol{r}$. This gives

$$\begin{aligned} \boldsymbol{\nabla} \cdot \overleftrightarrow{T} &= [\hat{\boldsymbol{x}}_i \partial_i] \cdot [T_{jk}\, \hat{\boldsymbol{x}}_j \hat{\boldsymbol{x}}_k] \\ &= [\partial_i T_{jk}]\; \hat{\boldsymbol{x}}_i \cdot (\hat{\boldsymbol{x}}_j \hat{\boldsymbol{x}}_k) \\ &= [\partial_i T_{jk}]\; \underbrace{(\hat{\boldsymbol{x}}_i \cdot \hat{\boldsymbol{x}}_j)}_{\delta_{ij}} \hat{\boldsymbol{x}}_k \\ &= [\partial_i T_{ik}]\; \hat{\boldsymbol{x}}_k. \end{aligned} \tag{4}$$

Hence, component i of the vector $\boldsymbol{\nabla} \cdot \overleftrightarrow{T}$ becomes

$$\left[\boldsymbol{\nabla} \cdot \overleftrightarrow{T}\right]_i = \partial_j T_{ji}. \tag{5}$$

b) We get

$$\begin{aligned}
\boldsymbol{v} \times \overleftrightarrow{T} &= [v_k \hat{\boldsymbol{x}}_k] \times [T_{ij}\, \hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j] \\
&= [v_k T_{ij}]\; \hat{\boldsymbol{x}}_k \times (\hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j) \\
&= [v_k T_{ij}]\; \underbrace{(\hat{\boldsymbol{x}}_k \times \hat{\boldsymbol{x}}_i)}_{\varepsilon_{ki\ell}\hat{\boldsymbol{x}}_\ell} \hat{\boldsymbol{x}}_j \\
&= \varepsilon_{ki\ell} v_k T_{ij} \hat{\boldsymbol{x}}_\ell \hat{\boldsymbol{x}}_j \\
&= \varepsilon_{k\ell i} v_k T_{\ell j} \hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j
\end{aligned} \tag{6}$$

where in the last step we renamed the dummy indices i and ℓ, in order to read off the ij component of $\boldsymbol{v} \times \overleftrightarrow{T}$ as what multiplies $\hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j$. This gives

$$(\boldsymbol{v} \times \overleftrightarrow{T})_{ij} = \varepsilon_{ik\ell} v_k T_{\ell j} \tag{7}$$

(here we also used $\varepsilon_{k\ell i} = \varepsilon_{ik\ell}$ which follows from the invariance of the Levi-Civita symbol under cyclic permutations of indices).

Similar manipulations give

$$\begin{aligned}
\overleftrightarrow{T} \times \boldsymbol{v} &= [T_{ij} \hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j] \times [v_k \hat{\boldsymbol{x}}_k] \\
&= [T_{ij} v_k](\hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j) \times \hat{\boldsymbol{x}}_k \\
&= [T_{ij} v_k] \hat{\boldsymbol{x}}_i \underbrace{(\hat{\boldsymbol{x}}_j \times \hat{\boldsymbol{x}}_k)}_{\varepsilon_{jk\ell}\hat{\boldsymbol{x}}_\ell} \\
&= \varepsilon_{jk\ell} T_{ij} v_k \hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_\ell \\
&= \varepsilon_{\ell kj} T_{i\ell} v_k \hat{\boldsymbol{x}}_i \hat{\boldsymbol{x}}_j,
\end{aligned} \tag{8}$$

so that the ij component of $\overleftrightarrow{T} \times \boldsymbol{v}$ is

$$(\overleftrightarrow{T} \times \boldsymbol{v})_{ij} = \varepsilon_{jk\ell} T_{ik} v_\ell. \tag{9}$$

As an example of the relevance of this cross product between a rank-2 tensor and a vector, we note that the angular momentum current density (which was mentioned in the lectures in connection with a very brief discussion of conservation of angular momentum) is given by $\overleftrightarrow{T} \times \boldsymbol{r}$, where $\overleftrightarrow{T}$ is here the Maxwell stress tensor and $\boldsymbol{r}$ is the position vector.

Problem 1

(a) The divergence theorem states that

$$\int_\Omega d^3r\, \nabla\cdot\vec{v} = \oint_a d\vec{a}\cdot\vec{v}$$

where the closed surface a is the boundary of the volume Ω.

Now take $\vec{v}(\vec{r}) = f(\vec{r})\,\vec{c}$ where $\vec{c}$ is a constant vector

$$\Rightarrow \nabla\cdot\vec{v} = \nabla\cdot(f\vec{c}) = f\underbrace{(\nabla\cdot\vec{c})}_{=0} + \vec{c}\cdot(\nabla f) = \vec{c}\cdot(\nabla f)$$

Inserting this into the divergence theorem gives

LHS : $\displaystyle\int_\Omega d^3r\, \nabla\cdot\vec{v} = \int_\Omega d^3r\, \vec{c}\cdot\nabla f \underset{\vec{c}\text{ constant}}{=} \vec{c}\cdot\int_\Omega d^3r\, \nabla f,$

RHS : $\displaystyle\oint_a d\vec{a}\cdot\vec{v} = \oint_a d\vec{a}\cdot(f\vec{c}) \overset{\vec{c}\text{ constant}}{=} \vec{c}\cdot\oint_a d\vec{a}\, f,$

i.e. $$\vec{c}\cdot\int_\Omega d^3r\,\nabla f = \vec{c}\cdot\oint_a d\vec{a}\, f$$

As this is valid for an arbitrary constant $\vec{c}$, it is also valid for the three particular cases $\vec{c}=\hat{x}$, $\vec{c}=\hat{y}$, and $\vec{c}=\hat{z}$, which give, respectively,

$$\int_\Omega d^3r\, \partial_x f = \oint_a da_x\, f \qquad (1)$$

$$\int_\Omega d^3r\, \partial_y f = \oint_a da_y\, f \qquad (2)$$

$$\int_\Omega d^3r\, \partial_z f = \oint_a da_z f \qquad (3)$$

Eqs. (1) - (3) can be combined into the vector equation

$$\boxed{\int_\Omega d^3r\, \nabla f = \oint_a d\vec{a}\, f}$$

(b) (To save some writing, I consider $2m_i$ instead of m_i.) We have

$$2m_i = \int_\Omega d^3r \left[\vec{r} \times (\nabla \times \vec{M})\right]_i + \int_a da \left[\vec{r} \times (\vec{M} \times \hat{n})\right]_i$$

$$= \int_\Omega d^3r\, \varepsilon_{ijk}\, r_j\, (\nabla \times \vec{M})_k + \int_a da\, \varepsilon_{ijk}\, r_j\, (\vec{M} \times \hat{n})_k$$

$$= \int_\Omega d^3r\, \varepsilon_{ijk}\, r_j\, \varepsilon_{klm}\, \partial_l M_m$$

$$+ \int_a da\, \varepsilon_{ijk}\, r_j\, \varepsilon_{klm}\, M_l\, \hat{n}_m$$

$$= \underbrace{\varepsilon_{ijk}}_{\varepsilon_{kij}}\, \varepsilon_{klm} \left[\int_\Omega d^3r\, r_j\, \partial_l M_m + \int_a da\, r_j\, M_l\, \hat{n}_m\right]$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \left[\int_\Omega d^3r\, r_j\, \partial_l M_m + \int_a da\, r_j\, M_l\, \hat{n}_m\right]$$

$$= \int_\Omega d^3r \left[r_j\, \partial_i M_j - r_j\, \partial_j M_i\right]$$

$$+ \int_a d^3r \left[r_j\, M_i\, \hat{n}_j - r_j\, M_j\, \hat{n}_i\right]$$

(c) Next, use

$$\partial_i (r_j M_j) = r_j \partial_i M_j + M_j \partial_i r_j$$

$$\Rightarrow \quad r_j \partial_i M_j = \partial_i (r_j M_j) - M_j \partial_i r_j \qquad (*)$$

and

$$\partial_j (r_j M_i) = r_j \partial_j M_i + M_i \partial_j r_j$$

$$\Rightarrow \quad r_j \partial_j M_i = \partial_j (r_j M_i) - M_i \partial_j r_j \qquad (**)$$

Inserting $(*)$ and $(**)$ into the expression for $2m_i$ gives

$$2m_i = \int_\Omega d^3 r \left[\partial_i (r_j M_j) - M_j \partial_i r_j - \partial_j (r_j M_i) + M_i \partial_j r_j \right] + \int_a da \left[r_j M_i \hat{n}_j - r_j M_j \hat{n}_i \right]$$

Noting that $\hat{n}_k \, da = da_k$, it follows from (a) that the integrals of the "red wiggly line" terms add to zero and that the integrals of the "green straight line" terms add to zero. Thus

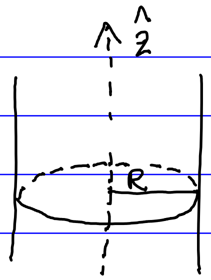
$$2m_i = \int_\Omega d^3 r \left[M_i \partial_j r_j - M_j \partial_i r_j \right]$$

Now using $\partial_i r_j = \delta_{ij}$ and $\partial_j r_j = \delta_{jj} = 3$

$$\Rightarrow 2m_i = \int_\Omega d^3 r \left[3M_i - M_j \delta_{ij} \right] = \int_\Omega d^3 r \left[3M_i - M_i \right]$$

$$= 2\int_\Omega d^3 r \, M_i \Rightarrow m_i = \int_\Omega d^3 r \, M_i \Rightarrow \boxed{\vec{m} = \int_\Omega d^3 r \, \vec{M}}$$

Problem 2



Magnetization inside the cylinder: $\vec{M} = ks\,\hat{z}$

($\vec{M} = 0$ outside the cylinder)

Using cylindrical coordinates we can write

$$\vec{M} = M_s\,\hat{s} + M_\varphi\,\hat{\varphi} + M_z\,\hat{z}$$

$\Rightarrow$ $M_s = M_\varphi = 0$, $M_z = ks$ (inside the cylinder)

(a) Volume bound current density:

$$\vec{J}_b = \nabla \times \vec{M} = -\frac{\partial M_z}{\partial s}\,\hat{\varphi}$$ (used expression for curl in cylindrical coords; all other terms vanish)

$$= -k\,\hat{\varphi}$$

Surface bound current density:

$$\vec{K}_b = \vec{M} \times \hat{n} = (kR\,\hat{z}) \times \hat{s} = kR\,\hat{\varphi}$$

As the bound currents are in the $\hat{\varphi}$ direction, they can be regarded as making up a continuous set of infinitely long solenoids, concentric with the cylinder and with different radii between 0 and R. Thus to find the magnetic field due to the bound currents, we can use that the magnetic field of a solenoid is

$$\begin{cases} \mu_0 n I\,\hat{z} & \text{inside the solenoid} \\ 0 & \text{outside the solenoid} \end{cases}$$

where I is the current carried by the wire in the solenoid and n is the # of turns per unit length

It immediately follows that $\vec{B} = 0$ outside the cylinder $(s > R)$ (since any point with $s > R$ is outside all the fictitious solenoids)

To find $\vec{B}$ inside the cylinder we need to find the field from the "individual" solenoids and then add up the contributions.

Let us first consider the contribution from the surface current density $\vec{K}_b = kR\,\hat{\varphi}$. The current passing through a height dh is $|\vec{K}_b|\,dh = kR\,dh$. Interpreted as a solenoid current, this gives

$$kR\,dh = I\,n\,dh \Rightarrow kR = nI$$

Thus the surface current gives the field

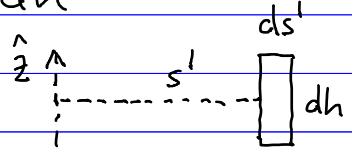
$$\vec{B}_{surface} = \mu_0 kR\,\hat{z}$$ inside the cylinder (i.e. for all $s < R$)

Next we consider the contribution $\vec{B}_{volume}$ from the bound volume current density. $\vec{B}_{volume}$ will depend on s since only the solenoids with radius $> s$ will contribute (as only for these solenoids is a point with coordinate s inside the solenoid).

Considering a solenoid with radius s' $(> s)$ the current passing through a small rectangle at s' with thickness ds' and height dh is

$$|\vec{j}_b|\,ds'\,dh = k\,ds'\,dh \equiv nI\,dh$$

$$\Rightarrow nI = k\,ds'$$



Thus the solenoid formula gives that the contribution from the solenoid at s' is

$$d\vec{B}_{volume} = -\mu_0 k \, ds' \, \hat{z}$$

(minus sign because the current is in the $-\hat{\varphi}$ direction)

Next we add the contributions from all the solenoids (with radii s' between s and R):

$$\vec{B}_{volume}(s) = \int_{s'=s}^{s'=R} d\vec{B}_{volume} = -\mu_0 k \hat{z} \int_s^R ds'$$

$$= -\mu_0 k (R-s) \hat{z}$$

Finally, the total magnetic field inside is found by adding the surface and volume contributions:

$$\underline{\underline{\vec{B}(s)}} = \vec{B}_{surface} + \vec{B}_{volume}(s)$$

$$= \mu_0 k R \hat{z} - \mu_0 k (R-s) \hat{z} = \mu_0 k s \hat{z} = \underline{\underline{\mu_0 \vec{M}}}$$

(b) The $\vec{H}$-field satisfies

$$\nabla \times \vec{H} = \vec{j}_f$$

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}$$

Since $\vec{M} = \begin{cases} ks\,\hat{z} & \text{for } s < R \\ 0 & \text{for } s > R \end{cases}$

we can write $\vec{M} = \Theta(R-s)\, ks\, \hat{z}$

where Θ is the Heaviside step function. Thus

$$\vec{M} = M_s\,\hat{s} + M_\varphi\,\hat{\varphi} + M_z\,\hat{z}$$

with $M_s = M_\varphi = 0$, $M_z = \Theta(R-s)\,ks$

$$\Rightarrow \quad \nabla\cdot\vec{M} = \frac{1}{s}\frac{\partial}{\partial s}(sM_s) + \frac{1}{s}\frac{\partial M_\varphi}{\partial\varphi} + \frac{\partial M_z}{\partial z} = 0+0+0=0$$

Thus $\nabla\cdot\vec{H}=0$, so in this problem $\vec{H}$ satisfies

$$\nabla\times\vec{H} = \vec{j}_f$$

$$\nabla\cdot\vec{H} = 0$$

But this is just like the equations for $\vec{B}$ in magnetostatics

$$\nabla\times\vec{B} = \mu_0\vec{j}$$

$$\nabla\cdot\vec{B} = 0$$

except that for $\vec{H}$, $\vec{j}_f$ plays the role that $\mu_0\vec{j}$ does for $\vec{B}$. So we can use intuition developed for $\vec{B}$ to find $\vec{H}$. Here it is very simple: since $\vec{j}_f=0$, $\vec{H}$ must be 0, since in magnetostatics, $\vec{B}=0$ in the absence of currents $\vec{j}$.

Thus the magnetic field in this problem becomes

$$\underline{\underline{\vec{B}}} = \mu_0(\vec{H}+\vec{M}) = \underline{\underline{\mu_0\vec{M}}} = \begin{cases}\mu_0 ks\,\hat{z} & \text{inside cylinder}\\ 0 & \text{outside cylinder}\end{cases}$$

in agreement with the more laborious calculation in (a)