TFY4240
Problem set 9

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Problem 1.
Consider a monochromatic plane electromagnetic wave that is traveling in the z direction and is linearly polarized in the x direction (i.e. $\boldsymbol{E}$ is parallel to the x axis).

a) Find all elements of the Maxwell stress tensor.

b) Verify that the differential version of the momentum conservation law is satisfied, by explicitly calculating both sides of the equation.

c) Consider a box-shaped volume Ω with lengths L_x, L_y, and L_z. Verify that the integral version of the momentum conservation law pertaining to Ω is satisfied, again by explicitly calculating both sides of the equation.

Problem 2.
In the lectures we considered the reflection and transmission of an electromagnetic wave at the flat interface between two different simple nonconducting media, with the incident wave polarized in the plane of incidence (so-called *p-polarization*). Actually, we cheated a little bit in the derivation of the Fresnel equations by assuming without justification that also the reflected and transmitted waves were polarized in the plane of incidence. In a more careful treatment one would deduce the polarization of the reflected and transmitted waves from the boundary conditions. This is the purpose of the following problem. For simplicity, we restrict our analysis here to normal incidence ($\theta_I = 0$).

We use the same coordinates as in Griffiths/lectures. Thus consider an incident wave propagating in the $+z$ direction (giving $\theta_R = \theta_T = \theta_I = 0$) and polarized in the x direction. The waves are transverse, so the polarization of the reflected and transmitted waves must be in the xy plane. Assuming linear polarization with the polarization directions of the reflected and transmitted waves rotated by angles φ_R and φ_T with respect to the x axis, one can write their polarization vectors as

$$\hat{\boldsymbol{n}}_R = \hat{\boldsymbol{x}} \cos\varphi_R + \hat{\boldsymbol{y}} \sin\varphi_R \quad \text{and} \quad \hat{\boldsymbol{n}}_T = \hat{\boldsymbol{x}} \cos\varphi_T + \hat{\boldsymbol{y}} \sin\varphi_T. \tag{1}$$

Use the boundary conditions on the electromagnetic fields to show that $\varphi_R = \varphi_T = 0$, i.e. the reflected and transmitted waves are also polarized in the x direction.

Problem 3.
When a wave passes from a medium with refractive index n_1 to one with index $n_2 < n_1$, Snell's law predicts that the transmitted angle θ_T will reach $\pi/2$ for a critical value θ_c of the incident angle, given by

$$\sin\theta_c = \frac{n_2}{n_1}. \tag{2}$$

For an incident angle $\theta_I > \theta_c$, the quantity θ_T loses its simple geometric meaning as an angle of refraction, which makes it necessary to give special consideration to the mathematical analysis and physical interpretation of this case. Snell's law gives $\sin\theta_T = (n_1/n_2)\sin\theta_I > 1$, so that $\cos\theta_T = \sqrt{1-\sin^2\theta_T} = i\sqrt{\sin^2\theta_T - 1}$, an imaginary quantity.

a) Show that in this case

$$\tilde{\boldsymbol{E}}_T = \tilde{\boldsymbol{E}}_{0T} e^{-\kappa z} e^{i(kx-\omega t)}, \tag{3}$$

where

$$\kappa \equiv \frac{\omega}{c} n_1 \sqrt{\sin^2\theta_I - \sin^2\theta_c} \quad \text{and} \quad k \equiv \frac{\omega}{c} n_1 \sin\theta_I. \tag{4}$$

This is a wave that propagates in the x direction (i.e. parallel to the interface) and is attenuated in the z direction.

b) Noting that the quantity $\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I}$ is now imaginary, show from the Fresnel equation for $\tilde{E}_{0R}$ that the reflection coefficient $R = 1$. (Hint: Show that $\tilde{E}_{0R}/\tilde{E}_{0I}$ can be written as a complex number of unit modulus.)

c) Construct the time-averaged Poynting vector $\langle \boldsymbol{S}_T \rangle$ for the transmitted wave, and show that $\langle \boldsymbol{S}_T \rangle \cdot \hat{\boldsymbol{z}} = 0$, so that on average no energy is transmitted in the z direction, giving transmission coefficient $T = 0$. (Hint: Use the formula $\langle \boldsymbol{S}_T \rangle = \frac{1}{2\mu}\mathrm{Re}[\tilde{\boldsymbol{E}}_T^* \times \tilde{\boldsymbol{B}}_T]$. The triple product rule $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$ may be useful, as well as $(\boldsymbol{k}_T\cdot\tilde{\boldsymbol{E}}_T)^* = 0$ which follows from $\nabla\cdot\tilde{\boldsymbol{E}}_T = 0$. Here $\boldsymbol{k}_T = k_{Tx}\hat{\boldsymbol{x}} + k_{Tz}\hat{\boldsymbol{z}} = k\hat{\boldsymbol{x}} + i\kappa\hat{\boldsymbol{z}}$.)

In view of the fact that $R = 1$, $T = 0$ this phenomenon is called **total internal reflection**.