TFY4240

Problem set 10

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Problem 1.

Consider a TE_{mn} mode propagating down a hollow metallic waveguide of rectangular cross section, as discussed in the lectures and Griffiths (Secs. 9.5.1 and 9.5.2). We want to calculate the "energy velocity" v_E, which is the velocity at which energy is transported down the waveguide. This can be defined as

$$v_E = \frac{\int da\ \langle \boldsymbol{S} \rangle \cdot \hat{\boldsymbol{z}}}{\int da\ \langle u \rangle} \tag{1}$$

where $\langle \boldsymbol{S} \rangle$ and $\langle u \rangle$ are the time-averaged Poynting vector and energy density of the wave, respectively, and the integral is over the rectangular cross section. The definition (1) is the natural generalization of the result $v = |\langle \boldsymbol{S} \rangle|/\langle u \rangle$ found for the propagation of an EM plane wave in a linear dielectric (including the vacuum with $v = c$). In that case $v = v_p = v_g$, i.e. the phase and group velocities coincide, due to the relation $\omega(k) = vk$. That is no longer the case for the waveguide, for which the confinement instead leads to $\omega(k) = \sqrt{(ck)^2 + \omega_{nm}^2}$, giving $v_p > c$ and $v_g < c$. Also note that the reason why the integrals $\int da\ \ldots$ are needed in (1) is because in the waveguide the confinement makes the integrands depend on the transverse coordinates x and y, unlike the case for a plane wave propagating in an infinite dielectric.

The goal of this problem is to show that for the waveguide, $v_E = v_g$. In the calculations it will be convenient to use the complex notation discussed in Problem 9.11 in Griffiths.

a) Show that

$$\int da\ \langle u \rangle = \frac{1}{2\mu_0} \frac{(\omega/c)^2}{(\omega/c)^2 - k^2} \int da |b_z|^2, \tag{2}$$

where b_z (called B_z by Griffiths) is the z-component of the complex amplitude of $\tilde{\boldsymbol{B}}$. Hint: Separate $\langle u \rangle$ into transverse (x, y) and longitudinal (z) contributions. Use Eqs. (9.180) in Griffiths to rewrite everything in terms of b_z and its derivatives. Use integration by parts, the boundary condition $B_\perp = 0$, and the wave equation for b_z to show that

$$\int da\ (\nabla_\perp b_z) \cdot (\nabla_\perp b_z^*) = -\int da\ b_z^* \nabla_\perp^2 b_z = [(\omega/c)^2 - k^2] \int da\ |b_z|^2 \tag{3}$$

where $\nabla_\perp \equiv \hat{\boldsymbol{x}}\partial_x + \hat{\boldsymbol{y}}\partial_y$.

b) Show that

$$\int da\ \langle \boldsymbol{S} \rangle \cdot \hat{\boldsymbol{z}} = \frac{1}{2\mu_0} \frac{\omega k}{(\omega/c)^2 - k^2} \int da\ |b_z|^2. \tag{4}$$

c) Find the energy velocity v_E and show that it equals the group velocity v_g.

Problem 2.

In the lectures we considered the Green function for the Laplacian ∇^2 in an infinite volume, defined by the differential equation[1]

$$\nabla^2 G(\boldsymbol{r}, \boldsymbol{r}') = -\delta(\boldsymbol{r} - \boldsymbol{r}') \tag{5}$$

and the boundary condition

$$G(\boldsymbol{r}, \boldsymbol{r}') \to 0 \text{ as } |\boldsymbol{r}| \to \infty \text{ (for an arbitrary fixed } \boldsymbol{r}'). \tag{6}$$

We used the fact that

$$\nabla^2 \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = -4\pi\delta(\boldsymbol{r} - \boldsymbol{r}') \tag{7}$$

to deduce that

$$G(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{8}$$

You may have found this "derivation" somewhat unsatisfactory since it required knowledge of the result (7); what if we hadn't known this beforehand? In this problem we therefore consider an alternative way of deriving (8).

a) Show that (5) can be rewritten as

$$\nabla^2_{\boldsymbol{R}} G = -\delta(\boldsymbol{R}) \tag{9}$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$ and $\nabla_{\boldsymbol{R}} \equiv \hat{\boldsymbol{x}}_i \frac{\partial}{\partial R_i}$. Thus we see that G is a function of $\boldsymbol{r}$ and $\boldsymbol{r}'$ only via the combination $\boldsymbol{R}$, so we write $G = G(\boldsymbol{R})$. The boundary condition (6) can then be rewritten as

$$G(\boldsymbol{R}) \to 0 \text{ as } |\boldsymbol{R}| \to \infty. \tag{10}$$

b) We will find $G(\boldsymbol{R})$ by the method of Fourier transformation. To this end, we express $G(\boldsymbol{R})$ as

$$G(\boldsymbol{R}) = \frac{1}{(2\pi)^3} \int d^3k \, g(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{R}}. \tag{11}$$

By also making use of the Fourier transform of $\delta(\boldsymbol{R})$, show that

$$g(\boldsymbol{k}) = \frac{1}{k^2}. \tag{12}$$

c) Use this to show that

$$G(\boldsymbol{R}) = \frac{1}{4\pi R}, \tag{13}$$

which thus gives the result (8).

[1] Here $\delta(\boldsymbol{r} - \boldsymbol{r}')$ is the Dirac delta function in 3 dimensions, defined as $\delta(\boldsymbol{r} - \boldsymbol{r}') = \delta(x - x')\delta(y - y')\delta(z - z')$. Note that we are here using the same name (δ) about two different functions (both the 1-dimensional and 3-dimensional delta function). Strictly speaking this is dubious notation (it would have been more appropriate to define the 3-dimensional delta function as $\delta^3(\boldsymbol{r} - \boldsymbol{r}')$, as is sometimes done), but this convention of letting the argument(s) tell us which function is meant is rather common in physics.

Problem 3.

In this problem we will use Fourier transformation (and contour integration) to calculate the Green function for the d'Alembertian, which satisfies the differential equation

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] G(\boldsymbol{r}, t; \boldsymbol{r}', t') = -\delta(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t'), \tag{14}$$

which by a change of variables to $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$ and $T = t - t'$ can be rewritten as

$$\left[\nabla_{\boldsymbol{R}}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial T^2}\right] G = -\delta(\boldsymbol{R})\delta(T), \tag{15}$$

so G is a function of $\boldsymbol{r}$, $\boldsymbol{r}'$, t and t' only through the combinations $\boldsymbol{R}$ and T. We therefore write $G = G(\boldsymbol{R}, T)$.

a) By expressing G in terms of its Fourier transform,

$$G(\boldsymbol{R}, T) = \frac{1}{(2\pi)^4}\int d^3k \int d\omega \; g(\boldsymbol{k}, \omega) e^{i(\boldsymbol{k}\cdot\boldsymbol{R} - \omega t)} \tag{16}$$

and also using the Fourier transform of the delta functions, show that

$$g(\boldsymbol{k}, \omega) = \frac{1}{k^2 - (\omega/c)^2}. \tag{17}$$

b) Viewed as a function of ω, $g(\boldsymbol{k}, \omega)$ has simple poles at $\omega = \pm kc$. When doing the inverse Fourier transform to find $G(\boldsymbol{R}, T)$, the answer will depend on how we integrate around those singularities. Here we are interested in the **retarded** Green function which satisfies $G = 0$ for $T < 0$. This condition will be satisfied if the integration path passes above the singularities (this is accomplished by replacing $\omega \to \omega + i\eta$ where η is a real positive infinitesimal, as this pushes the poles slightly below the real axis, so that by integrating along the real axis the integration path passes above the poles).

Use contour integration (the residue theorem) to show that the procedure just described indeed gives $G = 0$ for $T < 0$.

c) Use the same method to calculate G for $T > 0$. Show that this gives the result

$$G(\boldsymbol{R}, T) = \frac{1}{4\pi R}\delta(T - R/c). \tag{18}$$

In terms of the original variables, the retarded Green function is therefore

$$G(\boldsymbol{r}, t; \boldsymbol{r}', t') = \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|}\delta(t - t' - |\boldsymbol{r} - \boldsymbol{r}'|/c), \tag{19}$$

as also shown in the lectures using a completely different approach.

Problem 4.

Problem 9.19 in Griffiths.