

Solution to tutorial 12, TFY4240, 2015

Problem 1

Radiation fields from an electric dipole with dipole moment $\vec{p}$ and time dependence $e^{-i\omega t}$ (complex notation):

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{\omega^2}{c} \frac{1}{r} (\hat{r} \times \vec{p}) e^{i(kr - \omega t)}$$

$$\vec{E} = c \vec{B} \times \hat{r}$$

In this problem we can take the rotating electric dipole as the superposition of two dipoles labeled 1 and 2:

Dipole 1: dipole moment $\vec{p}_1 \equiv p_0 \hat{x}$, time dependence $e^{-i\omega t}$

Dipole 2: dipole moment $\vec{p}_2 \equiv p_0 \hat{y}$, time dep. $e^{-i(\omega t - \pi/2)} = i e^{-i\omega t}$

$$\text{since } \text{Re } e^{-i(\omega t - \frac{\pi}{2})} = \cos(\omega t - \frac{\pi}{2}) = \sin \omega t$$

$$\Rightarrow \vec{B} = \vec{B}_1 + \vec{B}_2$$

$$= \frac{\mu_0}{4\pi} \frac{\omega^2}{c} \frac{1}{r} e^{i(kr - \omega t)} \hat{r} \times (\vec{p}_1 + i \vec{p}_2)$$

$$\vec{E} = c \vec{B} \times \hat{r}$$

The physical (real) fields are obtained by taking the real part of these expressions.

Time-averaged Poynting vector:

$$\langle \vec{S} \rangle = \frac{1}{2\mu_0} \text{Re} [\vec{E} \times \vec{B}^*]$$

$$\begin{aligned}
 &= \frac{C}{2\mu_0} \operatorname{Re} \left[(\vec{B} \times \hat{r}) \times \vec{B}^* \right] = -\frac{C}{2\mu_0} \operatorname{Re} \left[\vec{B}^* \times (\vec{B} \times \hat{r}) \right] \\
 &= -\frac{C}{2\mu_0} \operatorname{Re} \left[\vec{B} \underbrace{(\vec{B}^* \cdot \hat{r})}_{=0} - \hat{r} (\vec{B} \cdot \vec{B}^*) \right] \\
 &= \frac{C}{2\mu_0} \operatorname{Re} |\vec{B}|^2 \hat{r} = \frac{C}{2\mu_0} |\vec{B}|^2 \hat{r}
 \end{aligned}$$

where

$$|\vec{B}|^2 = \vec{B}^* \cdot \vec{B} = \left(\frac{\mu_0}{4\pi} \frac{\omega^2}{c} \frac{1}{r} \right)^2 \underbrace{e^{-i(kr - \omega t)} e^{i(kr - \omega t)}}_{=1}$$

$$\cdot \left[\hat{r} \times (\vec{p}_1 - i\vec{p}_2) \right] \cdot \left[\hat{r} \times (\vec{p}_1 + i\vec{p}_2) \right]$$

$$= \left(\frac{\mu_0}{4\pi} \frac{\omega^2}{cr} \right)^2 \left\{ (\hat{r} \times \vec{p}_1)^2 + \overbrace{(-i)i}^{=1} (\hat{r} \times \vec{p}_2)^2 \right. \\ \left. \underbrace{-i(\vec{r} \times \vec{p}_2) \cdot (\vec{r} \times \vec{p}_1) + i(\vec{r} \times \vec{p}_1) \cdot (\hat{r} \times \vec{p}_2)}_{=0} \right\}$$

$$\Rightarrow \langle \vec{S} \rangle = \frac{\mu_0 \omega^4}{32\pi^2 c r^2} \left[(\hat{r} \times \vec{p}_1)^2 + (\hat{r} \times \vec{p}_2)^2 \right] \hat{r}$$

$$\Rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle = \langle \vec{S} \rangle \cdot \hat{r} r^2 = \frac{\mu_0 \omega^4}{32\pi^2 c} [(\hat{r} \times \vec{p}_1)^2 + (\hat{r} \times \vec{p}_2)^2]$$

We see that $\langle \vec{S} \rangle$ and $\langle dP/d\Omega \rangle$ are simply a sum of contributions from each dipole. This happens despite the fact that $\langle \vec{S} \rangle$ and $\langle dP/d\Omega \rangle$ are not linear but quadratic in fields. But in this case the cross terms $\propto (\vec{r} \times \vec{p}_1) \cdot (\vec{r} \times \vec{p}_2)$ vanish because the two dipoles have a $\pi/2$ phase difference. Since the contributions from the dipoles here add, the total radiated power is simply twice the radiated power from a single dipole,

i.e. the total power is

$$P = 2 \cdot \frac{\mu_0 P_0^2 \omega^4}{12\pi c} = \underline{\underline{\frac{\mu_0 P_0^2 \omega^4}{6\pi c}}}$$

Next let us consider the angular radiation pattern more explicitly. We have

$$\hat{r} \times \vec{p}_1 = \frac{1}{r} (x\hat{x} + y\hat{y} + z\hat{z}) \times p_0\hat{x} = \frac{p_0}{r} (-y\hat{z} + z\hat{y})$$

$$\Rightarrow (\hat{r} \times \vec{p}_1)^2 = \left(\frac{p_0}{r}\right)^2 (y^2 + z^2 - 2yz \underbrace{\hat{z} \cdot \hat{y}}_{=0}) = \left(\frac{p_0}{r}\right)^2 (y^2 + z^2)$$

$$\hat{r} \times \vec{p}_2 = \frac{1}{r} (x\hat{x} + y\hat{y} + z\hat{z}) \times p_0\hat{y} = \frac{p_0}{r} (x\hat{z} - z\hat{x})$$

$$\Rightarrow (\hat{r} \times \vec{p}_2)^2 = \left(\frac{p_0}{r}\right)^2 (x^2 + z^2 - 2xz \underbrace{\hat{z} \cdot \hat{x}}_{=0}) = \left(\frac{p_0}{r}\right)^2 (x^2 + z^2)$$

$$\Rightarrow (\hat{r} \times \vec{p}_1)^2 + (\hat{r} \times \vec{p}_2)^2 = \left(\frac{p_0}{r}\right)^2 (x^2 + y^2 + z^2)$$

Now using $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$, $z = r \cos\theta$

$$\Rightarrow (\hat{r} \times \vec{p}_1)^2 + (\hat{r} \times \vec{p}_2)^2 = p_0^2 [\sin^2\theta (\cos^2\varphi + \sin^2\varphi) + 2\cos^2\theta]$$

$$= p_0^2 [1 - \cos^2\theta + 2\cos^2\theta] = p_0^2 [1 + \cos^2\theta]$$

$$\Rightarrow \underline{\underline{\langle \frac{dP}{d\Omega} \rangle = \frac{\mu_0 P_0^2 \omega^4}{32\pi^2 c} (1 + \cos^2\theta)}}$$

See that $\langle dP/d\Omega \rangle$ is:

- independent of φ
- nonzero in all directions
- maximal for $\theta = 0, \pi$,
minimal for $\theta = \pi/2$

Problem 2

For $v \ll c$, the radiated power P is given by the Larmor formula,

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

For constant acceleration $a < 0$, the time t it takes to reach zero velocity from starting velocity v_0 is

$$t = \frac{v_0}{-a}$$

The radiated energy during this time is

$$\Delta E = Pt = \frac{\mu_0 q^2 a^2}{6\pi c} \frac{v_0}{(-a)} = \frac{\mu_0 q^2 v_0 (-a)}{6\pi c}$$

If the electron moves a distance d during this time,

$$-v_0^2 = 2ad \Rightarrow a = -\frac{v_0^2}{2d}$$

$$\Rightarrow \Delta E = \frac{\mu_0 q^2 v_0^3}{12\pi c d}$$

Compared to the electron's initial kinetic energy $E_0 = \frac{1}{2}mv_0^2$, the energy lost to radiation corresponds to a fraction

$$f \equiv \frac{\Delta E}{E_0} = \frac{\mu_0 q^2 v_0}{6\pi m c d} \\ = \frac{4\pi \cdot 10^{-7} \cdot (1.6 \cdot 10^{-19})^2 \cdot 10^5}{6\pi \cdot 9.11 \cdot 10^{-31} \cdot 3 \cdot 10^8 \cdot 30 \cdot 10^{-10}} = 0.002 \cdot 10^{-7} = \underline{\underline{2 \cdot 10^{-10}}}$$

Thus the fraction of energy lost to radiation is very small.

Problem 3

For this case with $\vec{v} \parallel \vec{a}$,

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5} \quad \left(\beta = \frac{v}{c}\right)$$

To find the angle of maximum radiation, we solve the equation

$$\frac{d}{d\theta} \frac{dP}{d\Omega} = 0$$

$$\Rightarrow \frac{2 \sin \theta \cos \theta}{(1 - \beta \cos \theta)^5} - 5 \frac{\beta \sin \theta}{(1 - \beta \cos \theta)^6} \cdot \sin^2 \theta = 0$$

Multiplying with $(1 - \beta \cos \theta)^6$ gives

$$2(1 - \beta \cos \theta) \sin \theta \cos \theta - 5\beta \sin^3 \theta = 0$$

The solutions $\theta = 0, \pi$ give angles of minimum radiation. These are not the angles of interest, so we may assume $\sin \theta \neq 0$ and divide by $\sin \theta$

$$\Rightarrow 2(1 - \beta \cos \theta) \cos \theta - 5\beta (1 - \cos^2 \theta) = 0$$

$$\Rightarrow 3\beta \cos^2 \theta + 2\cos \theta - 5\beta = 0$$

$$\begin{aligned} \Rightarrow \cos \theta &= \frac{-2 \pm \sqrt{4 - 4 \cdot 3\beta \cdot (-5\beta)}}{2 \cdot 3\beta} \\ &= \frac{-1 \pm \sqrt{1 + 15\beta^2}}{3\beta} \end{aligned}$$

Since $\cos \theta \geq -1$, only the + sign gives a valid solution. Since we are interested in the ultrarelativistic regime $\beta \simeq 1$, we define $\beta \equiv 1-x$ with $x \ll 1$, and rewrite $\cos \theta_{\max}$ in terms of x in a way suitable for Taylor expansion in x around $x=0$:

$$\begin{aligned}\cos \theta_{\max} &= \frac{\sqrt{1+15\beta^2}-1}{3\beta} = \frac{\sqrt{1+15(1-x)^2}-1}{3(1-x)} \\&= \frac{\sqrt{16-30x+15x^2}-1}{3(1-x)} = \frac{4\sqrt{1-\frac{15}{8}x+\frac{15}{16}x^2}-1}{3(1-x)} \\&\simeq \frac{4\left(1-\frac{1}{2}\frac{15}{8}x\right)-1}{3}(1+x) \quad (\text{correct to } O(x)) \\&= \left(1-\frac{5}{4}x\right)(1+x) \quad (\text{--- " ---}) \\&= 1-\frac{1}{4}x + O(x^2)\end{aligned}$$

$$\Rightarrow \cos \theta_{\max} \text{ is close to } 1 \Rightarrow \theta_{\max} \ll 1$$

$$\Rightarrow \cos \theta_{\max} \simeq 1 - \frac{1}{2} \theta_{\max}^2$$

$$\Rightarrow \theta_{\max}^2 \simeq \frac{1}{2}x \Rightarrow \underline{\theta_{\max} \simeq \sqrt{\frac{x}{2}}} = \underline{\sqrt{\frac{1-\beta}{2}}}$$

For an ultrarelativistic (u.r.) particle

$$\left(\frac{dP}{d\Omega}\right)_{\max, \text{u.r.}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{\sin^2 \theta_{\max}}{(1-\beta \cos \theta_{\max})^5}$$

For a particle instantaneously at rest,

$$\left(\frac{dP}{d\Omega}\right)_{\text{rest}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \sin^2 \theta$$

so the radiation is maximal at angle $\theta = \frac{\pi}{2}$, with

$$\left(\frac{dP}{d\Omega}\right)_{\text{max, rest}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}$$

This gives the ratio (assuming a is the same)

$$\frac{(dP/d\Omega)_{\text{max, u.r.}}}{(dP/d\Omega)_{\text{max, rest}}} = \frac{\sin^2 \theta_{\text{max}}}{(1 - \beta \cos \theta_{\text{max}})^5}$$

Here

$$\sin^2 \theta_{\text{max}} \approx \theta_{\text{max}}^2 \approx \frac{x}{2} + O(x^2),$$

$$1 - \beta \cos \theta_{\text{max}} \approx 1 - \beta \left(1 - \frac{x}{4}\right) = 1 - (1-x) \left(1 - \frac{x}{4}\right)$$

$$= 1 - (1-x) + \frac{x}{4} + O(x^2) = \frac{5x}{4} + O(x^2)$$

$$\Rightarrow \frac{(dP/d\Omega)_{\text{max, u.r.}}}{(dP/d\Omega)_{\text{max, rest}}} \approx \frac{\frac{x}{2}}{\left(\frac{5x}{4}\right)^5} = \frac{1}{2} \left(\frac{4}{5}\right)^5 \frac{1}{x^4} = \frac{512}{3125} \frac{1}{(1-\beta)^4}$$

To give the answer in terms of $\gamma = 1/\sqrt{1-\beta^2}$, we write

$$\gamma = \frac{1}{\sqrt{1-(1-x)^2}} = \frac{1}{\sqrt{1-(1-2x+x^2)}} = \frac{1}{\sqrt{2x-x^2}} \approx \frac{1}{\sqrt{2x}} = \frac{1}{\sqrt{2(1-\beta)}}$$

$$\Rightarrow \frac{(dP/d\Omega)_{\text{max, u.r.}}}{(dP/d\Omega)_{\text{max, rest}}} = \frac{512}{3125} (2\gamma^2)^4 \approx \underline{\underline{2.62 \gamma^8}}$$