Solution to tutorial 12, TFY4240

Problem 1

Radiation fields from an electric dipole with dipole moment $\vec{p}$ and time dependence $e^{-i\omega t}$ (complex notation):

$$\tilde{\vec{B}} = \frac{\mu_0}{4\pi}\frac{\omega^2}{c}\frac{1}{r}\left(\hat{r}\times\vec{p}\right) e^{i(kr-\omega t)}$$

$$\tilde{\vec{E}} = c\,\tilde{\vec{B}}\times\hat{r}$$

In this problem we can take the rotating electric dipole as the superposition of two dipoles labeled 1 and 2:

Dipole 1: dipole moment $\vec{p}_1 \equiv p_0\hat{x}$, time dependence $e^{-i\omega t}$

Dipole 2: dipole moment $\vec{p}_2 \equiv p_0\hat{y}$, time dep. $e^{-i(\omega t-\pi/2)} = i e^{-i\omega t}$

since $\mathrm{Re}\, e^{-i(\omega t-\frac{\pi}{2})} = \cos(\omega t - \frac{\pi}{2}) = \sin\omega t$

$$\Rightarrow \quad \tilde{\vec{B}} = \tilde{\vec{B}}_1 + \tilde{\vec{B}}_2$$

$$= \frac{\mu_0}{4\pi}\frac{\omega^2}{c}\frac{1}{r} e^{i(kr-\omega t)}\, \hat{r}\times\left(\vec{p}_1 + i\vec{p}_2\right)$$

$$\tilde{\vec{E}} = c\,\tilde{\vec{B}}\times\hat{r}$$

The physical (real) fields are obtained by taking the real part of these expressions.

Time-averaged Poynting vector:

$$\langle\vec{S}\rangle = \frac{1}{2\mu_0}\mathrm{Re}\left[\tilde{\vec{E}}\times\tilde{\vec{B}}^*\right]$$

$$= \frac{c}{2\mu_0} \mathrm{Re}\left[\left(\vec{\tilde{B}} \times \hat{r}\right) \times \vec{\tilde{B}}^*\right] = -\frac{c}{2\mu_0} \mathrm{Re}\left[\vec{\tilde{B}}^* \times \left(\vec{\tilde{B}} \times \hat{r}\right)\right]$$

$$= -\frac{c}{2\mu_0} \mathrm{Re}\left[\vec{\tilde{B}} \underbrace{\left(\vec{\tilde{B}}^* \cdot \hat{r}\right)}_{=0} - \hat{r}\left(\vec{\tilde{B}} \cdot \vec{\tilde{B}}^*\right)\right]$$

$$= \frac{c}{2\mu_0} \mathrm{Re} \left|\vec{\tilde{B}}\right|^2 \hat{r} = \frac{c}{2\mu_0} \left|\vec{\tilde{B}}\right|^2 \hat{r}$$

where

$$|\vec{\tilde{B}}|^2 = \vec{\tilde{B}}^* \cdot \vec{\tilde{B}} = \left(\frac{\mu_0}{4\pi} \frac{\omega^2}{c} \frac{1}{r}\right)^2 \underbrace{e^{-i(kr-\omega t)} e^{i(kr-\omega t)}}_{=1}$$

$$\cdot \left[\hat{r} \times (\vec{p}_1 - i\vec{p}_2)\right] \cdot \left[\hat{r} \times (\vec{p}_1 + i\vec{p}_2)\right]$$

$$= \left(\frac{\mu_0}{4\pi} \frac{\omega^2}{cr}\right)^2 \Big\{ \left(\hat{r} \times \vec{p}_1\right)^2 + \overbrace{(-i)i}^{=1} \left(\hat{r} \times \vec{p}_2\right)^2$$
$$\underbrace{- i \left(\vec{r} \times \vec{p}_2\right) \cdot \left(\vec{r} \times \vec{p}_1\right) + i \left(\vec{r} \times \vec{p}_1\right) \cdot \left(\hat{r} \times \vec{p}_2\right)}_{=0} \Big\}$$

$$\Rightarrow \langle \vec{S} \rangle = \frac{\mu_0 \omega^4}{32\pi^2 c r^2} \left[\left(\hat{r} \times \vec{p}_1\right)^2 + \left(\hat{r} \times \vec{p}_2\right)^2\right] \hat{r}$$

$$\Rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle = \langle \vec{S} \rangle \cdot \hat{r}\, r^2 = \frac{\mu_0 \omega^4}{32\pi^2 c} \left[\left(\hat{r} \times \vec{p}_1\right)^2 + \left(\hat{r} \times \vec{p}_2\right)^2\right]$$

We see that $\langle \vec{S} \rangle$ and $\langle dP/d\Omega \rangle$ are simply a sum of contributions from each dipole. This happens despite the fact that $\langle \vec{S} \rangle$ and $\langle dP/d\Omega \rangle$ are not linear but quadratic in fields. But in this case the cross terms $\propto (\vec{r} \times \vec{p}_1) \cdot (\vec{r} \times \vec{p}_2)$ vanish because the two dipoles have a $\pi/2$ phase difference. Since the contributions from the dipoles here add, the total radiated power is simply twice the radiated power from a single dipole, i.e. the total power is

$$P = 2 \cdot \frac{\mu_0 p_0^2 \omega^4}{12\pi c} = \underline{\underline{\frac{\mu_0 p_0^2 \omega^4}{6\pi c}}}$$

Next let us consider the angular radiation pattern more explicitly. We have

$$\hat{r} \times \vec{p}_1 = \frac{1}{r}(x\hat{x} + y\hat{y} + z\hat{z}) \times p_0 \hat{x} = \frac{p_0}{r}(-y\hat{z} + z\hat{y})$$

$$\Rightarrow (\hat{r} \times \vec{p}_1)^2 = \left(\frac{p_0}{r}\right)^2 (y^2 + z^2 - 2yz \underbrace{\hat{z}\cdot\hat{y}}_{=0}) = \left(\frac{p_0}{r}\right)^2 (y^2 + z^2)$$

$$\hat{r} \times \vec{p}_2 = \frac{1}{r}(x\hat{x} + y\hat{y} + z\hat{z}) \times p_0 \hat{y} = \frac{p_0}{r}(x\hat{z} - z\hat{x})$$

$$\Rightarrow (\hat{r} \times \vec{p}_2) = \left(\frac{p_0}{r}\right)^2 (x^2 + z^2 - 2xz \underbrace{\hat{z}\cdot\hat{x}}_{=0}) = \left(\frac{p_0}{r}\right)^2 (x^2 + z^2)$$

$$\Rightarrow (\hat{r} \times \vec{p}_1)^2 + (\hat{r} \times \vec{p}_2)^2 = \left(\frac{p_0}{r}\right)^2 (x^2 + y^2 + 2z^2)$$

Now using $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$

$$\Rightarrow (\hat{r} \times \vec{p}_1)^2 + (\hat{r} \times \vec{p}_2)^2 = p_0^2 \left[\sin^2\theta(\cos^2\varphi + \sin^2\varphi) + 2\cos^2\theta\right]$$

$$= p_0^2\left[1 - \cos^2\theta + 2\cos^2\theta\right] = p_0^2\left[1 + \cos^2\theta\right]$$

$$\Rightarrow \underline{\underline{\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}(1 + \cos^2\theta)}}$$

See that $\langle dP/d\Omega \rangle$ is:
- independent of φ
- nonzero in all directions
- maximal for $\theta = 0, \pi$, minimal for $\theta = \pi/2$

Problem 2

For $v << c$, the radiated power P is given by the Larmor formula,

$$P = \frac{\mu_0 q^2 a^2}{6\pi c}$$

For constant acceleration $a < 0$, the time t it takes to reach zero velocity from starting velocity v_0 is

$$t = \frac{v_0}{-a}$$

The radiated energy during this time is

$$\Delta E = Pt = \frac{\mu_0 q^2 a^2}{6\pi c} \frac{v_0}{(-a)} = \frac{\mu_0 q^2 v_0 (-a)}{6\pi c}$$

If the electron moves a distance d during this time,

$$-v_0^2 = 2ad \Rightarrow a = -\frac{v_0^2}{2d}$$

$$\Rightarrow \Delta E = \frac{\mu_0 q^2 v_0^3}{12\pi c d}$$

Compared to the electron's initial kinetic energy $E_0 = \frac{1}{2} m v_0^2$, the energy lost to radiation corresponds to a fraction

$$f \equiv \frac{\Delta E}{E_0} = \frac{\mu_0 q^2 v_0}{6\pi m c d}$$

$$= \frac{4\cancel{\pi} \cdot 10^{-7} \cdot (1.6 \cdot 10^{-19})^2 \cdot 10^5}{6\cancel{\pi} \cdot 9.11 \cdot 10^{-31} \cdot 3 \cdot 10^8 \cdot 30 \cdot 10^{-10}} = 0.002 \cdot 10^{-7} = \underline{\underline{2 \cdot 10^{-10}}}$$

Thus the fraction of energy lost to radiation is very small.

Problem 3

For this case with $\vec{v} \parallel \vec{a}$,

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \qquad \left(\beta \equiv \frac{v}{c}\right)$$

To find the angle of maximum radiation, we solve the equation

$$\frac{d}{d\theta}\frac{dP}{d\Omega} = 0$$

$$\Rightarrow \quad \frac{2\sin\theta\cos\theta}{(1-\beta\cos\theta)^5} - 5\frac{\beta\sin\theta}{(1-\beta\cos\theta)^6}\cdot\sin^2\theta = 0$$

Multiplying with $(1-\beta\cos\theta)^6$ gives

$$2(1-\beta\cos\theta)\sin\theta\cos\theta - 5\beta\sin^3\theta = 0$$

The solutions $\theta = 0, \pi$ give angles of minimum radiation. These are not the angles of interest, so we may assume $\sin\theta \neq 0$ and divide by $\sin\theta$

$$\Rightarrow \quad 2(1-\beta\cos\theta)\cos\theta - 5\beta(1-\cos^2\theta) = 0$$

$$\Rightarrow \quad 3\beta\cos^2\theta + 2\cos\theta - 5\beta = 0$$

$$\Rightarrow \quad \cos\theta = \frac{-2 \pm \sqrt{4 - 4\cdot 3\beta\cdot(-5\beta)}}{2\cdot 3\beta}$$

$$= \frac{-1 \pm \sqrt{1+15\beta^2}}{3\beta}$$

Since $\cos\theta \geq -1$, only the + sign gives a valid solution. Since we are interested in the ultrarelativistic regime $\beta \simeq 1$, we define $\beta \equiv 1-x$ with $x \ll 1$, and rewrite $\cos\theta_{max}$ in terms of x in a way suitable for Taylor expansion in x around $x=0$:

$$\cos\theta_{max} = \frac{\sqrt{1+15\beta^2}-1}{3\beta} = \frac{\sqrt{1+15(1-x)^2}-1}{3(1-x)}$$

$$= \frac{\sqrt{16-30x+15x^2}-1}{3(1-x)} = \frac{4\sqrt{1-\frac{15}{8}x+\frac{15}{16}x^2}-1}{3(1-x)}$$

$$\simeq \frac{4\left(1-\frac{1}{2}\frac{15}{8}x\right)-1}{3}(1+x) \qquad \text{(correct to } O(x))$$

$$= \left(1-\frac{5}{4}x\right)(1+x) \qquad (\text{— " —})$$

$$= 1-\frac{1}{4}x + O(x^2)$$

$\Rightarrow \cos\theta_{max}$ is close to 1 $\Rightarrow \theta_{max} \ll 1$

$\Rightarrow \cos\theta_{max} \simeq 1-\frac{1}{2}\theta_{max}^2$

$$\Rightarrow \theta_{max}^2 \simeq \frac{1}{2}x \Rightarrow \underline{\underline{\theta_{max}}} \simeq \sqrt{\frac{x}{2}} = \underline{\underline{\sqrt{\frac{1-\beta}{2}}}}$$

For an ultrarelativistic (u.r.) particle

$$\left(\frac{dP}{d\Omega}\right)_{max,\,u.r.} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{\sin^2\theta_{max}}{(1-\beta\cos\theta_{max})^5}$$

For a particle instantaneously at rest,

$$\left(\frac{dP}{d\Omega}\right)_{rest} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \sin^2\theta$$

so the radiation is maximal at angle $\theta = \frac{\pi}{2}$, with

$$\left(\frac{dP}{d\Omega}\right)_{max, rest} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}$$

This gives the ratio (assuming a is the same)

$$\frac{(dP/d\Omega)_{max, u.r.}}{(dP/d\Omega)_{max, rest}} = \frac{\sin^2\theta_{max}}{(1-\beta\cos\theta_{max})^5}$$

Here

$$\sin^2\theta_{max} \simeq \theta_{max}^2 \simeq \frac{x}{2} + O(x^2),$$

$$1-\beta\cos\theta_{max} \simeq 1-\beta\left(1-\frac{x}{4}\right) = 1-(1-x)\left(1-\frac{x}{4}\right)$$

$$= 1-(1-x) + \frac{x}{4} + O(x^2) = \frac{5x}{4} + O(x^2)$$

$$\Rightarrow \frac{(dP/d\Omega)_{max, u.r.}}{(dP/d\Omega)_{max, rest}} \simeq \frac{\frac{x}{2}}{\left(\frac{5x}{4}\right)^5} = \frac{1}{2}\left(\frac{4}{5}\right)^5 \frac{1}{x^4} = \frac{512}{3125}\frac{1}{(1-\beta)^4}$$

To give the answer in terms of $\gamma = 1/\sqrt{1-\beta^2}$, we write

$$\gamma = \frac{1}{\sqrt{1-(1-x)^2}} = \frac{1}{\sqrt{1-(1-2x+x^2)}} = \frac{1}{\sqrt{2x-x^2}} \simeq \frac{1}{\sqrt{2x}} = \frac{1}{\sqrt{2(1-\beta)}}$$

$$\Rightarrow \underline{\underline{\frac{(dP/d\Omega)_{max, u.r.}}{(dP/d\Omega)_{max, rest}}}} = \frac{512}{3125}(2\gamma^2)^4 \simeq \underline{\underline{2.62\,\gamma^8}}$$